1. Work out the character table for $\Sigma_5$. (H7 Q1 and H8 Q3 should give you a lot to start with, and you can deduce the missing entries as we have done for $\Sigma_4$.)

2. Let $V_{\text{Std}}$ be the standard representation of $\Sigma_5$. Decompose

(a) $\text{Sym}^2(V_{\text{Std}})$, (NOTE: Don’t forget H6 Q1(a).)
(b) $V_{\text{Std}} \otimes V_{\text{Std}}$, and
(c) $V_{\text{Std}} \otimes V_{\text{Std}} \otimes V_{\text{Std}}$

into irreducible representations. (This means you should say how many times each irreducible appears, and not that you should provide an explicit decomposition into those irreducibles.)

3. As in H8 Q1, let $G$ be a finite group, $X$ a finite set with $G$-action, $(V, \rho)$ the corresponding permutation representation of $G$, and $\chi_V$ its character.

(a) Show that $\chi_V(g) = \chi_V(g^{-1})$ for all $g \in G$ (this will be used in (b) below).

An action of $G$ on a set $X$ is said to be transitive if for any $x_1, x_2 \in X$, there is at least one $g \in G$ with $g \cdot x_1 = x_2$. The action is said to be doubly transitive if it is transitive, and if for any $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq y_1$ and $x_2 \neq y_2$ there is at least one $g \in G$ with $g \cdot x_1 = x_2$ and $g \cdot y_1 = y_2$.

Suppose that $X$ is a set with at least two elements, and that $G$ acts transitively on $X$. Since this means that there is only one orbit, by H8 Q2(b) this means that we can write the permutation representation $V$ as $V = V_{\text{Triv}} \oplus V'$, where $V_{\text{Triv}}$ is the trivial representation.

(b) Under these conditions show that the following are equivalent:

(b1) The action of $G$ on $X$ is doubly transitive
(b2) The action of $G$ on $X \times X$ has only two orbits: the diagonal and its complement.
(b3) $\langle \chi_V^2, \chi_{\text{Triv}} \rangle = 2$.
(b4) The representation $V'$ is irreducible.

In (b2) $G$ acts on $X \times X$ by the ‘diagonal action’, namely $g \cdot (x, y) = (g \cdot x, g \cdot y)$. 