1. By Green's theorem, one way to find the area of a region in $\mathbb{R}^2$ is to integrate the vector field $\mathbf{F} = (-\frac{y}{2}, \frac{x}{2})$ around the outside of the boundary.

We already know a parameterization of the hypocycloid $\mathbf{c}$ (from Homework 6, # 2):

$$\begin{aligned} x(\theta) &= \cos^3(\theta) & (x'(\theta), y'(\theta)) &= (-3\cos^2(\theta)\sin(\theta), 3\cos(\theta)\sin^2(\theta)) \\ y(\theta) &= \sin^3(\theta) & \mathbf{F} &= (-\tfrac{\sin^3(\theta)}{2}, \tfrac{\cos^3(\theta)}{2}) \end{aligned}$$

$$\text{for } 0 \le \theta \le 2\pi.$$

The dot product of $\mathbf{F}$ with the velocity vector of the parameterization is

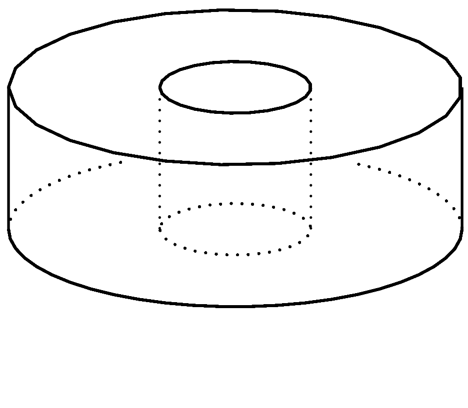
$$\begin{aligned} \mathbf{F} \cdot (x', y') &= \frac{3}{2}(\cos^2(\theta)\sin^4(\theta) + \cos^4(\theta)\sin^2(\theta)) \\ &= \frac{3}{2}\cos^2(\theta)\sin^2(\theta) \\ &= \frac{3}{8}\sin^2(2\theta) \end{aligned}$$

So the integral becomes

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds = \frac{3}{8}\int_0^{2\pi} \sin^2(2\theta) = \frac{3}{8}\pi$$

2.

(a) The region V is a cylinder of radius 3 with a smaller cylinder of radius 1 removed:



The boundary surface comes in four natural pieces: the top disk, the bottom disk, the outside cylinder, and the inside cylinder.

(b) Piece one: the top of the cylinder (oriented upwards)

$$\begin{array}{rclcrcl} x(r,\theta) & = & r\cos(\theta) & \quad & \mathbf{T}_r & = & (\cos(\theta), \sin(\theta), 0) \\ y(r,\theta) & = & r\sin(\theta) & & \mathbf{T}_\theta & = & (-r\sin(\theta), r\cos(\theta), 0) \\ z(r,\theta) & = & 2 & & \mathbf{N} & = & (0, 0, r) \end{array}$$

$$\text{for } 1 \le r \le 3,\ 0 \le \theta \le 2\pi.$$

The normal is oriented upwards as it should be. With this parameterization, we have $\mathbf{F} = (2r\cos(\theta), r^3\cos(\theta)\sin^2(\theta), 2r^2\sin(\theta)\cos(\theta))$, and

$$\mathbf{F} \cdot \mathbf{N} = 2r^3 \sin(\theta)\cos(\theta) = r^3 \sin(2\theta).$$

If S_1 is the top of the cylinder, this gives

$$\iint_{S_1} \mathbf{F} \cdot dS \;=\; \int_1^3 \int_0^{2\pi} r^3 \sin(2\theta)\, d\theta\, dr = 0.$$

Piece two: the bottom of the cylinder (oriented downwards)

$$\begin{array}{rclcrcl} x(r,\theta) & = & r\cos(\theta) & \quad & \mathbf{T}_r & = & (\cos(\theta), \sin(\theta), 0) \\ y(r,\theta) & = & r\sin(\theta) & & \mathbf{T}_\theta & = & (-r\sin(\theta), r\cos(\theta), 0) \\ z(r,\theta) & = & 0 & & \mathbf{N} & = & (0, 0, r) \end{array}$$

$$\text{for } 1 \le r \le 3,\ 0 \le \theta \le 2\pi.$$

We need to use the downwards oriented normal, or $\mathbf{N} = (0, 0, -r)$ (although in this case it doesn't make much difference). We have $\mathbf{F} = (2r\cos(\theta), r^3\cos(\theta)\sin^2(\theta), 0)$, and so $\mathbf{F} \cdot \mathbf{N} = 0$. If S_2 is the bottom part of the cylinder, this gives

$$\iint_{S_2} \mathbf{F} \cdot dS \;=\; 0.$$

Piece three: the outside surface, oriented outwards.

$$\begin{aligned}
x(v,\theta) &= 3\cos(\theta) & \mathbf{T}_\theta &= (-3\sin(\theta), 3\cos(\theta), 0)\\
y(v,\theta) &= 3\sin(\theta) & \mathbf{T}_v &= (0,0,1)\\
z(v,\theta) &= v & \mathbf{N} &= (3\cos(\theta), 3\sin(\theta), 0)
\end{aligned}$$

$$\text{for } 0 \leq v \leq 2,\ 0 \leq \theta \leq 2\pi.$$

The normal vector is pointing outwards, as it should be. We have

$$\mathbf{F} = (6\cos(\theta), 27\cos(\theta)\sin^2(\theta), 9v\cos(\theta)\sin(\theta)),$$

with $\mathbf{F}\cdot\mathbf{N} = 18\cos(\theta) + 81\cos(\theta)\sin^3(\theta)$.

If S_3 is the outside surface, this gives

$$\iint_{S_3} \mathbf{F}\cdot dS \;=\; \int_0^2 \int_0^{2\pi} 18\cos^2(\theta) + 81\cos(\theta)\sin^3(\theta)\, d\theta\, dv = 36\pi$$

Piece four: the inside surface, oriented inwards.

$$\begin{aligned}
x(v,\theta) &= \cos(\theta) & \mathbf{T}_\theta &= (-\sin(\theta), \cos(\theta), 0)\\
y(v,\theta) &= \sin(\theta) & \mathbf{T}_v &= (0,0,1)\\
z(v,\theta) &= v & \mathbf{N} &= (\cos(\theta), \sin(\theta), 0)
\end{aligned}$$

$$\text{for } 0 \leq v \leq 2,\ 0 \leq \theta \leq 2\pi.$$

Here the normal is pointing outwards, and we need to use $\mathbf{N} = (-\cos(\theta), -\sin(\theta), 0)$ instead. We have

$$\mathbf{F} = (2\cos(\theta), \cos(\theta)\sin^2(\theta), v\cos(\theta)\sin(\theta)),$$

with $\mathbf{F}\cdot\mathbf{N} = -2\cos^2(\theta) - \cos(\theta)\sin^3(\theta)$.

If S_4 is the inside surface, this gives

$$\iint_{S_4} \mathbf{F}\cdot dS \;=\; \int_0^2 \int_0^{2\pi} -2\cos^2(\theta) - \cos(\theta)\sin^3(\theta)\, d\theta\, dv = -4\pi.$$

Therefore, if S is the oriented boundary surface made up of S_1, S_2, S_3, and S_4 together, we have

$$\iint_S = 0 + 0 + 36\pi - 4\pi = 32\pi.$$

(c) $\mathrm{Div}(\mathbf{F}) = 2 + 3xy$. If V is the cylinder (with the smaller cylinder removed) then the integral of xy over V is zero by symmetry, and so the integral of $\mathrm{Div}(\mathbf{F})$ over V is the same as the integral of 2 over V, i.e., twice the volume of V, so

$$\iiint_V \mathrm{Div}(\mathbf{F})\ dV = 2(\text{volume of } V) = 2(2)(3^2 - 1)\pi = 32\pi.$$

3. We're starting with $\mathbf{F}(x, y, z) = (y, z, x^2)$ on $\mathbb{R}^3$.

(a) We can use the usual parameterization of the unit sphere:

$$\begin{aligned} x(\phi,\theta) &= \sin(\phi)\cos(\theta) & \mathbf{T}_\phi &= (\cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi)) \\ y(\phi,\theta) &= \sin(\phi)\sin(\theta) & \mathbf{T}_\theta &= (-\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0) \\ z(\phi,\theta) &= \cos(\phi) & \mathbf{N} &= (\sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)) \end{aligned}$$

$$\text{for } 0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi.$$

The normal vector is oriented outwards. In this parameterization the vector field becomes $\mathbf{F} = (\sin(\phi)\sin(\theta), \cos(\phi), \sin^2(\phi)\cos^2(\theta))$, and the dot product of the vector field and normal vector is

$$\mathbf{F}\cdot\mathbf{N} = \sin^3(\phi)\sin(\theta)\cos(\theta) + \sin^2(\phi)\cos(\phi)\sin(\theta) + \sin^3(\phi)\cos(\phi)\sin^2(\theta).$$

Note that only the last term : $\sin^3(\phi)\cos(\phi)\sin^2(\theta)$ will contribute anything to the integral, since (by integrating by θ) first, the other terms give zero.

The integral becomes

$$\iint_{S_1} \mathbf{F}\cdot dS \ = \ \int_0^{2\pi}\int_0^{\pi/2} \sin^3(\phi)\cos(\phi)\sin^2(\theta)\, d\phi\, d\theta = \frac{\pi}{4}$$

(b) Parameterizing S_2:

$$\begin{aligned} x(r,\theta) &= r\cos(\theta) & \mathbf{T}_r &= (\cos(\theta),\sin(\theta),0) \\ y(r,\theta) &= r\sin(\theta) & \mathbf{T}_\theta &= (-r\sin(\theta), r\cos(\theta), 0) \\ z(r,\theta) &= 0 & \mathbf{N} &= (0,0,r) \end{aligned}$$

$$\text{for } 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

The normal is pointing upwards. In this parameterization the vector field becomes $\mathbf{F} = (r\sin(\theta), 0, r^2\cos^2(\theta))$, and the dot product with the normal is

$$\mathbf{F}\cdot\mathbf{N} = r^3\cos^2(\theta).$$

Therefore we have

$$\iint_{S_2} \mathbf{F}\cdot dS = \int_0^1\int_0^{2\pi} r^3\cos^2(\theta)\,d\theta\,dr = \frac{\pi}{4}.$$

(c) The vector field $\mathbf{G} = (\frac{z^2}{2}, \frac{x^3}{3}, \frac{y^2}{2})$ is one of many with $\mathrm{Curl}(\mathbf{G}) = \mathbf{F}$.

(d) Parameterizing the unit circle $\mathbf{c}$:

$$\begin{aligned} x(\theta) &= \cos(\theta) & (x'(\theta), y'(\theta), z'(\theta)) &= (-\sin(\theta), \cos(\theta), 0) \\ y(\theta) &= \sin(\theta) & \mathbf{G} &= (0, \tfrac{\cos^3(\theta)}{3}, \tfrac{\sin^2(\theta)}{2}) \\ z(\theta) &= 0 \end{aligned}$$

$$\text{for } 0 \le \theta \le 2\pi.$$

The dot product of $\mathbf{G}$ with the velocity vector is $\mathbf{G}\cdot(x',y',z') = \frac{\cos^4(\theta)}{3}$, giving

$$\begin{aligned} \int_{\mathbf{c}} \mathbf{G}\cdot ds &= \frac{1}{3}\int_0^{2\pi} \cos^4(\theta)\,d\theta \\ &= \left(\frac{1}{12}\cos^3(\theta)\sin(\theta) + \frac{1}{8}\cos(\theta)\sin(\theta) + \frac{1}{8}\theta\right)_{\theta=0}^{\theta=2\pi} \\ &= \frac{\pi}{4}. \end{aligned}$$

(e) Explanation one:

If $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and $\text{Div}(\mathbf{F}) = 0$ then there is some vector field $\mathbf{G}$ with $\text{Curl}(\mathbf{G}) = \mathbf{F}$. So if S_1 and S_2 are two oriented surfaces with the same oriented boundary curve $\mathbf{c}$, then

$$\iint_{S_1} \mathbf{F} \cdot dS = \int_{\mathbf{c}} \mathbf{G} \cdot ds = \iint_{S_2} \mathbf{F} \cdot dS,$$

where the two equalities are obtained by applying Stokes' theorem, and using $\text{Curl}(\mathbf{G}) = \mathbf{F}$.

Explanation two:

Let V be the volume enclosed by S_1 and S_2. The oriented boundary of V is S_1 and S_2 with opposite orientations. But then by the divergence theorem:

$$\iint_{S_1} \mathbf{F} \cdot dS - \iint_{S_2} \mathbf{F} \cdot dS = \iiint_V \text{Div}(\mathbf{F})\, dV = 0,$$

so again the two flux integrals are equal.

4. We're starting with the vector field $\mathbf{F}(x, y, z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$, which is defined on $\mathbb{R}^3$ minus the z-axis.

(a) For $\mathbf{c}_1$ the unit circle in the xy-plane (we can use the parameterization from 3(d)):

$$\begin{aligned} x(\theta) &= \cos(\theta) \\ y(\theta) &= \sin(\theta) \\ z(\theta) &= 0 \end{aligned} \qquad \begin{aligned} (x'(\theta), y'(\theta), z'(\theta)) &= (-\sin(\theta), \cos(\theta), 0) \\ \mathbf{F} &= (-\sin(\theta), \cos(\theta), 0) \end{aligned}$$

for $0 \leq \theta \leq 2\pi$.

The dot product of vector field and velocity vector is $\mathbf{F} \cdot (x', y', z') = \sin^2(\theta) + \cos^2(\theta) = 1$, giving

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot ds \quad = \quad \int_0^{2\pi} 1\, d\theta = 2\pi.$$

(b) The calculation for the unit circle $\mathbf{c}_2$ lifted to $z = 3$ is almost identical:

$$\begin{array}{rclcrcl} x(\theta) & = & \cos(\theta) & \quad & (x'(\theta), y'(\theta), z'(\theta)) & = & (-\sin(\theta), \cos(\theta), 0) \\ y(\theta) & = & \sin(\theta) & & \mathbf{F} & = & (-\sin(\theta), \cos(\theta), 0) \\ z(\theta) & = & 3 & & & & \end{array}$$

$$\text{for } 0 \leq \theta \leq 2\pi.$$

The dot product of vector field and velocity vector is again $\mathbf{F} \cdot (x', y', z') = 1$, and so

$$\int_{\mathbf{c}_2} \mathbf{F} \cdot ds \quad = \quad \int_0^{2\pi} 1 \, d\theta = 2\pi.$$

(c) Shifting the unit circle in the x-direction by 3 gives us

$$\begin{array}{rclcrcl} x(\theta) & = & \cos(\theta) + 3 & \quad & (x'(\theta), y'(\theta), z'(\theta)) & = & (-\sin(\theta), \cos(\theta), 0) \\ y(\theta) & = & \sin(\theta) & & \mathbf{F} & = & (-\frac{\sin(\theta)}{10+6\cos(\theta)}, \frac{\cos(\theta)+3}{10+6\cos(\theta)}, 0) \\ z(\theta) & = & 0 & & & & \end{array}$$

$$\text{for } 0 \leq \theta \leq 2\pi.$$

The dot product of vector field and velocity vector is

$$\mathbf{F} \cdot (x', y', z') = \frac{1 + 3\cos(\theta)}{10 + 6\cos(\theta)}.$$

The anti-derivative of $\frac{1+3\cos(\theta)}{10+6\cos(\theta)}$ is a bit hard to find (sorry – I didn't mean for it to be so difficult): an anti-derivative is $\arctan(\tan(\theta/2)) - \arctan(\tan(\theta/2)/2)$.

The fact that the anti-derivative is periodic gives

$$\begin{aligned} \int_{\mathbf{c}_3} \mathbf{F} \cdot ds &= \int_0^{2\pi} \frac{1 + 3\cos(\theta)}{10 + 6\cos(\theta)} \, d\theta \\ &= (\arctan(\tan(\theta/2)) - \arctan(\tan(\theta/2)/2))_{\theta=0}^{\theta=2\pi} = 0. \end{aligned}$$

(d)

$$\mathrm{Curl}(\mathbf{F}) = \left(0, 0, \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right) = (0,0,0).$$

To show that $\int_{\mathbf{c}_3} \mathbf{F} \cdot ds = 0$, let S_3 be the disk with $\mathbf{c}_3$ as a boundary oriented upwards, then Stokes' theorem gives

$$\int_{\mathbf{c}_3} \mathbf{F} \cdot ds = \iint_{S_3} \mathrm{Curl}(\mathbf{F}) \cdot dS = 0.$$

To show that the answers in (a) and (b) should be the same, let S_{12} be the cylinder $x^2+y^2 = 1$, $0 \le z \le 3$, oriented outwards. This circle has $\mathbf{c}_1$ and $\mathbf{c}_2$ as boundary curves, but to get the orientation compatible with the orientation on S_{12}, we have to travel around $\mathbf{c}_2$ backwards. So, in this case Stokes' theorem gives

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot ds - \int_{\mathbf{c}_2} \mathbf{F} \cdot ds = \iint_{S_{12}} \mathrm{Curl}(\mathbf{F}) \cdot dS = 0,$$

and so $\int_{\mathbf{c}_1} \mathbf{F} \cdot ds = \int_{\mathbf{c}_2} \mathbf{F} \cdot ds$.

(e) Any surface S with boundary $\mathbf{c}_1$ or $\mathbf{c}_2$ would have to cross the z-axis. Since $\mathbf{F}$ isn't defined on the z-axis, we can't apply Stokes' theorem to integrate over S.

5. The vector field

$$\mathbf{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right),$$

is defined everywhere but the origin.

(a) If S is any closed surface not containing the origin, and V the region that it surrounds, then we can apply the divergence theorem to compute $\iint_S \mathbf{F} \cdot dS$:

$$\iint_S \mathbf{F} \cdot dS = \iiint_V \mathrm{Div}(\mathbf{F})\, dV = 0,$$

since $\mathrm{Div}(\mathbf{F}) = 0$.

(b) Now suppose that S is any closed surface that surrounds the origin. We can't apply the divergence theorem to the region inside S since $\mathbf{F}$ is not defined at the origin.

But, if we let S_ϵ be a small sphere of radius ϵ around the origin (small enough to be contained inside of S), and V the region *between* S_ϵ and S, then we can apply the divergence theorem to V.

If we orient S outwards, and also orient S_ϵ outwards, then the orientation on S is right for the divergence theorem, but S_ϵ should be reversed. Switching the sign, the divergence theorem then gives us:

$$\iint_S \mathbf{F} \cdot dS - \iint_{S_\epsilon} \mathbf{F} \cdot dS = \iiint_V \mathrm{Div}(\mathbf{F})\, dV = 0,$$

or $\iint_S \mathbf{F} \cdot dS = \iint_{S_\epsilon} \mathbf{F} \cdot dS$.

This shows that the value of $\iint_S \mathbf{F} \cdot dS$ doesn't depend on the surface S which surrounds the origin, since it's equal to the integral over any sufficiently small sphere surrounding the origin, and since we can always find a common such sphere for any two surfaces S and S' surrounding the origin.

The actual value of this flux integral over any sphere around the origin was computed in Homework 10 question 5 to be

$$\iint_{S_\epsilon} \mathbf{F} \cdot dS = 4\pi.$$