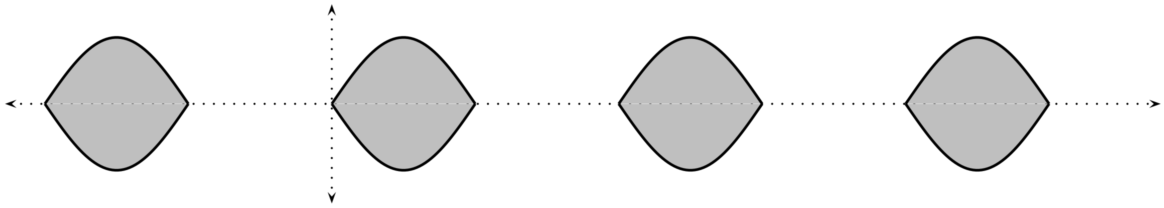
1. The region $U = \{(x,y) \mid |y| \le \sin(x)\}$, consists of infinitely many disconnected "lobes":



each one associated to an interval $x \in [2k\pi, (2k+1)\pi]$ on the x-axis for $k \in \mathbb{Z}$, i.e., those x values where $\sin(x) \ge 0$. The grey points (those points with $|y| < \sin(x)$) are the interior points, and the dark black points (those with $|y| = \sin(x)$) are the boundary points.

2. If $u(x,y,t) = e^{-2t}\sin(3x)\cos(2y)$, then

$$\begin{aligned} u_x(x,y,t) &= 3e^{-2t}\cos(3x)\cos(2y), \\ u_y(x,y,t) &= -2e^{-2t}\sin(3x)\sin(2y), \text{ and} \\ u_t(x,y,t) &= -2e^{-2t}\sin(3x)\cos(2y). \end{aligned}$$

If u is the height of a vibrating membrane (like a drum) above (x,y) at time t, then at a point (x_0,y_0) and time t_0, $u_t(x_0,y_0,t_0)$ represents how fast the membrane is moving up and down over top the point (x_0,y_0) at time t_0. For the other two derivatives, imagine the surface of the membrane at this frozen instant t_0, then $u_x(x_0,y_0,t_0)$ is the rate of change of the height of the graph going in the x-direction, and $u_y(x_0,y_0,t_0)$ the rate of change going on the y-direction.

3. If $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ is the function $\mathbf{F}(x,y) = \Big(\sin(\pi x)\cos(\pi y),\ ye^{xy},\ x^2+y^3\Big)$, then

$$\mathbf{DF}(x,y) = \begin{bmatrix} \pi\cos(\pi x)\cos(\pi y) & -\pi\sin(\pi x)\sin(\pi y) \\ y^2e^{xy} & (1+xy)e^{xy} \\ 2x & 3y^2 \end{bmatrix}, \text{ so } \mathbf{DF}(1,2) = \begin{bmatrix} -\pi & 0 \\ 4e^2 & 3e^2 \\ 2 & 12 \end{bmatrix}.$$

If we head in the direction $\vec{v} = (3,-2)$, this means that the instantaneous rate of change of the three functions is given by

$$\mathbf{DF}(1,1)\vec{v} = \begin{bmatrix} -\pi & 0 \\ 4e^2 & 3e^2 \\ 2 & 12 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3\pi \\ 6e^2 \\ -18 \end{bmatrix}.$$

In other words, if we're at the point $(1,2)$, and we head off in the direction $(3,-2)$, the instantaneous rate of change of $\sin(\pi x)\cos(\pi y)$ is -3π, the instantaneous rate of change of ye^{xy} is $6e^2$, and the instantaneous rate of change of x^2+y^3 is -18.

4. We're starting with the function

$$f(x,y) = \begin{cases} \dfrac{y^2x}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

(a) If we restrict to the x-axis, i.e. points of the form $(x,0)$, we get the function

$$f(x,0) = \begin{cases} \dfrac{0}{x^2+0^2} & \text{if } (x,0)\neq(0,0) \\ 0 & \text{if } (x,0)=(0,0) \end{cases}$$

which can be more succinctly described by simply saying that $f(x,0)=0$ for all values of x.

The partial derivative $f_x(0,0)$ certainly exists. According to the definition of $f_x(0,0)$ we only need to understand $f(x,0)$ and see if it has an x-derivative at $x=0$. The zero function (which is what $f(x,0)$ is) is certainly differentiable – any constant function is. Its derivative, like all constant functions, is 0. Thus $f_x(0,0)=0$.

We can also go directly to the definition of the partial derivative:

$$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0,$$

but it seems simpler just to think about the restriction as above.

(b) Restricting the function f to the y-axis, gives $f(0,y)=0$, just like the restriction to the x-axis. For the same reasons, $f_y(0,0)$ exists and is zero.

(c) If f were differentiable at $(0,0)$ then its derivative matrix would be

$$\mathbf{D}f(0,0) = \begin{bmatrix} \dfrac{\partial f}{\partial x}(0,0) & \dfrac{\partial f}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

(d) If f were differentiable at $(0,0)$ then the instantaneous rate of change in the direction $\vec{v}=(1,1)$ would be given by

$$\mathbf{D}f(0,0)\vec{v} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

(e) If we restrict f to points of the form (t,t), we get the function

$$f(t,t) = \begin{cases} \dfrac{t^3}{t^2+t^2} & \text{if } (t,t) \neq (0,0) \\ 0 & \text{if } (t,t) = (0,0) \end{cases}$$

or simply the function $f(t,t) = t/2$.

As a function of t, this is a line with slope $\frac{1}{2}$. Its rate of change at $t=0$ is therefore also $\frac{1}{2}$. You can also write this more formally: $\frac{d}{dt} f(t,t) = \frac{1}{2}$, and then plugging in $t=0$ gives $\frac{1}{2}$.

(f) If f were differentiable, the answers to (d) and (e) would be the same. A consequence of the definition of differentiability is that the matrix gives the instantaneous rate of change of f in all directions from $(0,0)$.

(g) Therefore, f is obviously not differentiable at $(0,0)$.

It's worthwhile thinking a bit more about the function f. To try and see what it looks like, let's restrict it to lines of the form $(v_x t,\, v_y t)$ for some nonzero vector $\vec{v} = (v_x, v_y)$:

$$f(v_x t,\, v_y t) = \begin{cases} \dfrac{v_x v_y^2 t^3}{(v_x^2+v_y^2)t^2} & \text{if } (v_x t,\, v_y t) \neq (0,0) \\ 0 & \text{if } (v_x t,\, v_y t) = (0,0) \end{cases}$$

or simply $f(v_x t,\, v_y t) = \left(\dfrac{v_x v_y^2}{v_x^2+v_y^2} \right) t.$

This shows us that the restriction of f to any direction $\vec{v}$ is just a linear function of t, with t-slope $\frac{v_x v_y^2}{v_x^2+v_y^2}$, which is zero along the x and y axes. Two views of the graph are shown below:

In any direction $\vec{v}$, the derivative exists and is equal to $\frac{v_x v_y^2}{v_x^2+v_y^2}$. Since this doesn't depend linearly on (v_x, v_y) it is clear that f is not differentiable at $(0,0)$. This was exactly the kind of example mentioned in the tutorial.

5. This time we're starting with the function $f(x, y) = 25 - x^2 - 2y^2$.

(a) $$\begin{array}{ll} & f(2,3) = 3 \\ f_x(x,y) = -2x & f_x(2,3) = -4 \\ f_y(x,y) = -4y & f_x(2,3) = -12 \end{array}$$

(b) $$\begin{array}{ll} g_x(x,y) = m & g_x(2,3) = m \\ g_y(x,y) = n & g_x(2,3) = n \end{array}$$

(c) Clearly we want $m = -4$ and $n = -12$. The graph of $-4x - 12y + c$ passes through $-4(2) - 12(3) + c = -44 + c$ over $(2, 3)$. In order for this to be 3 we need $c = 47$.

(d) $g(2 + t\,v_x,\, 3 + t\,v_y) = -4(2 + t\,v_x) - 12(3 + t\,v_y) + 47 = 3 + (-4v_x - 12v_y)t$. This has derivative $-4v_x - 12v_y$ at $t = 0$ (or for any t).

(e) $f(2 + t\,v_x,\, 3 + t\,v_y) = 3 + (-4v_x - 12v_y)t - (v_x^2 + 2v_y^2)t^2$. This also has derivative $-4v_x - 12v_y$ when $t = 0$.

(f) The fact that the answers to (d) and (e) are the same is certainly the behaviour we expect from a differentiable function – it's an indication that f is well approximated by its tangent plane over $(2, 3)$.

This behaviour (that the instantaneous rate of change along lines agrees with that of the plane determined by the partial derivatives) is not enough to guarantee that a function is actually differentiable, although it takes a bit of work to construct an example. (CHALLENGE: find one for yourself).

Our function f *is* differentiable though. For instance, the partial derivatives $f_x(x, y) = -2x$ and $f_y(x, y) = -4y$ are certainly continuous, and we know that's enough to imply the differentiability of f.

(g) Actually, although it may not immediately appear like it, the definition of what it means for $\mathbf{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ to be differentiable is exactly that the separate component functions F_1, F_2, and F_3 be individually differentiable. It is equivalent to the definition of differentiabilty given in the book.

(h) The derivative of a single function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at a point $(x_1, \ldots, x_n)$ is something which computes the instantaneous rate of change of f in any direction $\vec{v} = (v_1, \ldots, v_n)$, and this association is *linear* in $\vec{v}$. Therefore, the derivative $\mathbf{DF}$ of a function $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at a point $(x_1, \ldots, x_n)$ should be something which computes the m different instantaneous rates of change in a direction $\vec{v}$, and the dependence of the rates of change on $\vec{v}$ should again be linear. But a linear function from n-vectors to m-vectors is exactly given by an $(m \times n)$ matrix.