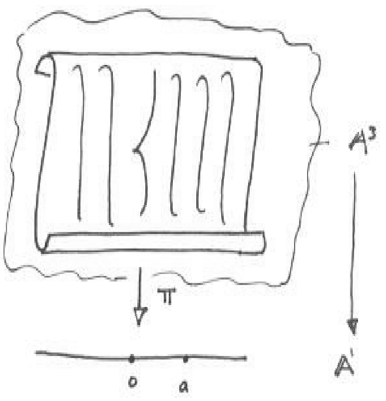
One of the most spectacular successes of Grothendieck's approach to algebraic geometry is the ability to put geometric and arithmetic phenomema on an equal footing.

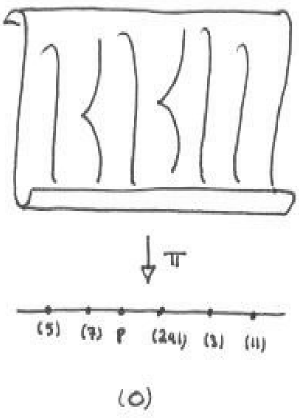
A good example is to compare the following complex surface mapping to a curve:

$$\begin{tikzcd} \{\, (x,y,t) \mid y^2 = x^3 + t \,\} \arrow[r, phantom, "\subseteq"] & \mathbb{A}^3_{x,y,t} \arrow[dd, "\pi"] \\ (x,y,t) \arrow[d, mapsto] & \\ t & \mathbb{A}^1_t \end{tikzcd}$$



(we get the fibres by "plugging in values of t", or "setting $t - a = 0$"), with the following arithmetic surface mapping to $\mathrm{Spec}(\mathbb{Z})$:

$$\begin{tikzcd} \mathrm{Spec}\left(\frac{\mathbb{Z}[x,y]}{y^2=x^3+N}\right) \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}$$



(we get the fibers by setting "$p = 0$"). The fibres are curves in characteristic p (except for one fibre in char. 0).

The construction joins the curves in different characterstics into one global object, and it's possible to compare and transfer geometric information among the fibres.

The purpose of this talk is to explain a wonderful theorem of Mori, whose proof uses this synthesis of geometry and arithmetic in a crucial way: The theorem and conclusion can be stated purely in terms of complex analytic projective manifolds, but the only known argument goes through characteristic p.

The theorem is this:

THEOREM (**Mori**) *Suppose that X is a smooth projective variety, such that $c_1(T_X) > 0$, then for every point x of X, there is a rational curve R in X passing through x with $c_1(T_X) \cdot R \leq \dim(X) + 1$.*

The interest in the theorem is that X is covered by rational curves (in fact more is true: X is *rationally connected*). The condition that $c_1(T_X) \leq \dim(X) + 1$ is a technical refinement which is important for the proof. What is actually necessary is that there be *some* bound on $c_1(T_X) \cdot R$, relatively independent of detailed geometric information about X.

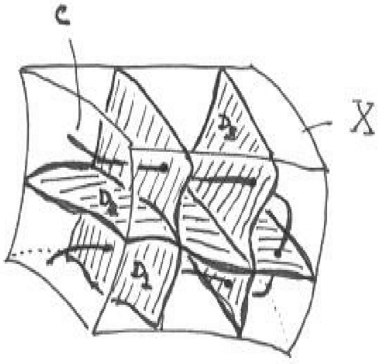
To explain the statement of the theorem, for any projective variety X there is a divisor (or really a divisor class) in X called $c_1(T_X)$,

$c_1(T_X) = \sum_i n_i D_i$, with $n_i \in \mathbb{Z}$,

and for any curve $C \subset X$,

$$c_1(T_X) \cdot C = \sum_i n_i (D_i \cdot C)$$

where $D_i \cdot C$ is the number of times that D_i intersects C.



We say that $c_1(T_X) > 0$ if $c_1(T_X) \cdot C > 0$ for all curves $C \subset X$.

In many cases, it is possible to verify that $c_1(T_X) > 0$ without testing all possible curves.

EXAMPLE: *If X is a hypersurface of degree d in $\mathbb{P}^n$ with $d \leq n$, then $c_1(T_X) > 0$. Slightly more generally, if X is a complete intersection in $\mathbb{P}^n$ of hypersurfaces of degrees $d_1, \ldots d_r$, with $d_1 + \cdots + d_r \leq d$, then $c_1(T_X) > 0$.*

Here is an example where both the hypothesis and conclusion of the theorem are easy to see

EXAMPLE: *If $X = \mathbb{P}^n$, then $c_1(T_X)$ is given by hypersurfaces of degree $(n+1)$. In particular, for any curve C of degree d, $c_1(T_X) \cdot C = (n+1)d > 0$ by Bezout's theorem.*

Through any point x if $\mathbb{P}^n$, we can *find a rational curve R through x (pick any line) with $c_1(T_X) \cdot R = (n+1) \leq \dim(\mathbb{P}^n) + 1$.*

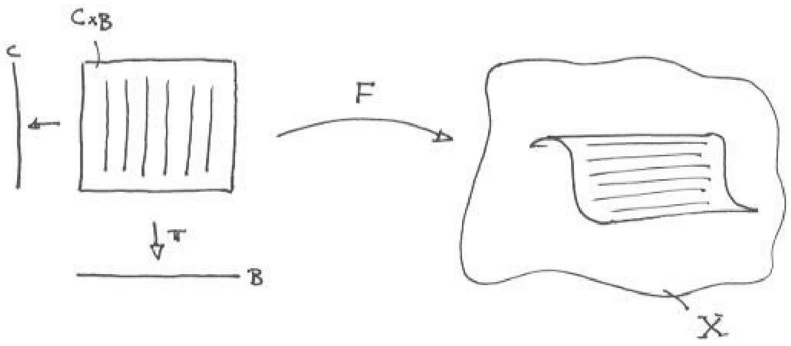
Before explaining the proof of the theorem we need to first talk about a lemma which will allow us to produce rational curves, as long as we have a "good" family of curves to start with.

First, a definition.

Definition *Suppose that we have:*

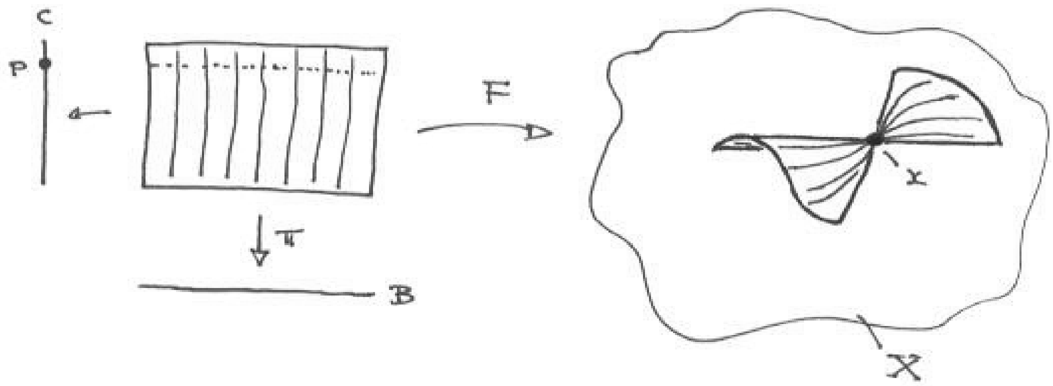
> *C – a smooth projective curve*
>
> *B – a smooth curve (affine or projective),*

Then we call a map $C \times B \xrightarrow{F} X$ a family of maps of C into X, over the base B.



We're interested in the following situation: there is a point p of C which is sent to the same point x of X under all of the maps in the family.

We also want the family to be *nontrivial*, that is, the family should actually move, and not send everything to a fixed point, or a fixed curve.

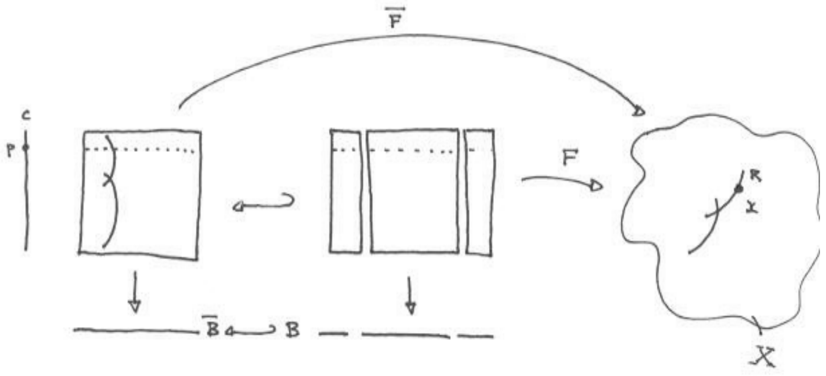


Mori's Breaking lemma has two forms:

> Form 1: If B is a projective curve, then such a family is impossible.

This doesn't seem much like a breaking lemma, or one which produces rational curves.

But, suppose we have such a family over an affine base B, and we compactify B to a smooth projective curve $B \subset \overline{B}$. Then, if we try and extend the map $F : C \times B \longrightarrow X$ to the compactification $\overline{F} : C \times \overline{B} \longrightarrow X$, we may be forced to blow up at a few points, in order to extend the map. In fact, form 1 implies that we *have* to blow up somewhere along $p \times B$ (otherwise we'd get an impossible family) and therefore the family "breaks" and we have a rational curve R passing through x.



> Form 2: If we find such a family over an affine curve B, then there is a rational curve R passing through x.

Form 1 implies form 2, as shown above. Here's a sketch of how to prove form 1:

Consider $C \times B$ (if B is projective) as a family of maps of B into X with base C. By hypothesis, there is a point $p \in C$ so that the whole fibre B_p gets sent to a single point in X.

Look at the set

$$S = \{q \in C \mid \text{the fibre } B_q \text{ above } q \text{ gets sent to a single point in } X\}.$$

Since $C \times B \longrightarrow C$ is proper, it follows that S is open. Since $C \times B \longrightarrow C$ is flat, it follows that S is closed. (both these statements require a little argument to see). By hypothesis, S is nonempty (it contains p), so we conclude that $S = C$.

In other words, this is a trivial family, contrary to assumption.

So – given a point $x \in X$, our goal is to find a curve $C \subset X$ (or more generally, a curve C along with a map $f : C \longrightarrow X$) passing through x such that C moves in a positive dimensional family with the point $x \in f(C)$ fixed. Then, by Mori's lemma, we'll get a rational curve passing through x.

The next ingredient is slightly technical – given a map $f : C \longrightarrow X$ we need to understand when C can move in a family (especially a family with one point fixed).

The result we need, from deformation theory, is this:

LEMMA *Given a map $f : C \longrightarrow X$ then the curve C moves in a family of dimension at* least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) \geq c_1(T_X) \cdot f[C] + (1 - g_C)\dim(X)$$

where g_C is the genus of C.

However complicated (or unexpected) the formula seems, it's important to note that the formula really only depends on the number $c_1(T_X) \cdot f[C]$ mentioned in the statement of the theorem.

The previous dimension is the dimension of all deformations of C (or at least a lower bound for this), but we want a little more – we want deformations of C with a point fixed.

It costs $\dim(X)$ conditions to fix a point on C, and so the final dimension bound is

$$(*) \qquad \dim\begin{pmatrix}\text{deformations of } C \text{ leaving a} \\ \text{point } p \text{ fixed}\end{pmatrix} \geq c_1(T_X) \cdot f[C] - g_C \dim(X)$$

Therefore, whenever the formula on the right side of (*) is positive, C does deform, and by the previous argument we can find a rational curve passing through x.

So – let's start with an X satisfying the hypothesis of our theorem:

$$c_1(T_X) \cdot f[C] > 0$$

for all curves C and all (nontrivial) maps $f : C \longrightarrow X$.

Pick a point $x \in X$, and let C be any curve passing through x.

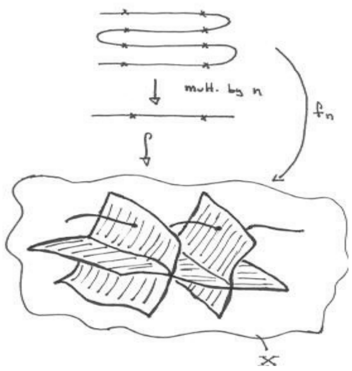
Here are some cases:

> Case 0: $g(C) = 0$, that is, C is *already* a rational curve. Here we don't need to do anything (we found a rational curve passing through x!). But note that since $C_1(T_X) \cdot C > 0$ (which is equal to formula (*) in this case), so that C moves in a family. By "breaking" C we can get a rational curve passing through x with a smaller intersection $c_1(T_X) \cdot R$. We can continue with this until we find a rational curve with $c_1(T_X) \cdot R \leq \dim(X) + 1$.

We'll use this case implicitly below – if we produce one rational curve passing through x we can assume there is a rational curve R satisfying the inequality $c_1(T_X) \cdot R \leq \dim(X) + 1$.

Case 1: $g(C) = 1$. In this case the hypothesis that $c_1(T_X) \cdot C > 0$ doesn't (immediately) imply that $c_1(T_X) \cdot C - \dim(X) > 0$. Here's how to fix that:

Every genus 1 curve C has a map "multiplication by n" from C to *itself* which is n^2 to 1.



Let f_n be the composite map

$$f_n : C \longrightarrow C \hookrightarrow X$$

Then $c_1(T_X) \cdot f_n[C] = n^2(c_1(T_X) \cdot C)$, so that by taking n large enough, we can ensure that

$$n^2 \Big(c_1(T_X) \cdot C \Big) - 1 \cdot \dim(X) > 0$$

and then we use our breaking argument again.

Case ≥ 2: $g(C) \geq 2$. Here again we have the problem that the hypothesis $c_1(T_X) \cdot C > 0$ doesn't imply what we need, that $(c_1(T_X) \cdot C - g \dim(X)) > 0$. We have another problem: there aren't any multiplication maps $C \longrightarrow C$ like before to "bump up" the intersection number.

We can try and take a ramified cover $C' \longrightarrow C$, but that increases both $c_1(T_X) \cdot C'$ and the negative part, the genus of C' at the same time, and unfortunately the negative part increases faster!

So it seems like (in general) that we're stuck.

But, there is a way out if we happen to be working over a finite field.

Over finite fields, there are *always* maps $C \longrightarrow C$ from of a curve to itself of positive degree: The Frobenius Map!

Concretely: If C is defined over a finite field $\mathbb{F}_q$, say as a subset of projective space $\mathbb{P}^m$, then the map

$$\begin{array}{ccc} C & \xrightarrow{F_q} & C \\ [y_0 : y_1 : \cdots : y_m] & \mapsto & [y_0^q : y_1^q : \cdots : y_m^q] \end{array}$$

is called the *Frobenius* map (the geometric one), and is a map of degree q from C to itself.

As before, we use this to bump up $c_1(T_X) \cdot f[C]$ until it is greater than $g_C \dim(X)$ and we're done by using the breaking argument again.

So, we can assume the theorem proven whenever X is defined over a finite field.

Going from characteristic p to characteristic 0:

Here's where we need to use the force of scheme theory.

Suppose that we have a variety X in characteristic zero.

X is defined by a finite number of equations in some projective space $\mathbb{P}^m$, and each equation involves finitely many coefficients. For any point $x \in X$, its description also involves only a finite of elements of the field (its coordinates).

Let A be the (finitely generated) $\mathbb{Z}$-algebra generated by these coefficients.

Example: *Suppose that $X \subset \mathbb{P}^4$ is defined by*

$$Z_0^4 + \sqrt{5}Z_1^4 + \pi Z_2^4 - Z_3^4 + Z_4^4 = 0$$

and x is the point $x = [1{:}0{:}0{:}0{:}i]$ then A is the ring generated by $\sqrt{5}$, π, i, (and ± 1)

$$A = \mathbb{Z}[\sqrt{5}, i, \pi] = \frac{\mathbb{Z}[y_1, y_2, y_3]}{(y_1^2 - 5, y_2^2 + 1)}.$$

Since all the cofficients needed to define X are in the ring A, we can consider the family $\mathcal{X}$ defined over $\mathrm{Spec}(A)$ by using the same equations in $\mathbb{P}^m_A$; this family has a section corresponding to x. We can also restrict to an open subset to ensure that all fibres are smooth schemes. If $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite field (since it is a finitely generated $\mathbb{Z}$-algebra, and since $\mathbb{Q}$ is not), so all the fibres over the closed points of $\mathrm{Spec}(A)$ are varieties over finite fields.

Here are two explanations of how to boost our finite field knowledge to characteristic zero (or other fields of characteristic p).

Fancy Way: Let $\mathcal{H}^d_A(\mathcal{X})$ be the relative Hilbert scheme of rational curves in $\mathcal{X}/\mathrm{Spec}(A)$ of degree $d \leq \dim(X) + 1$. We can further look at the subvariety of curves passing through the section corresponding to $x \in X$.

This variety maps *properly* to $\mathrm{Spec}(A)$. Since the image contains all maximal ideals (by our proof of the theorem over finite fields) of $\mathrm{Spec}(A)$, it must be surjective, and therefore contains even the characteristic zero fibres.

More Concrete Way: The question of whether or not there is a rational curve mapping into X (and passing through x) can be expressed as finding $(m+1)$ (if $X \subset \mathbb{P}^m$) polynomials of degree d, $P_0(W_0, W_1)$, $P_1(W_0, W_1)$,..., $P_m(W_0, W_1)$ whose coefficients satisfy some conditions (given by the requirement that the curve be contained in X, and pass through x).

The coefficients (thought of as unknowns) and their equations define a projective scheme over $\mathrm{Spec}(A)$. There is a component corresponding to the maps that map to a single point, the remaining components are still a projective scheme over $\mathrm{Spec}(A)$. Again, since this is a proper scheme over $\mathrm{Spec}(A)$, the image is closed. Since the image contains all maximal ideals, it must be all of $\mathrm{Spec}(A)$, and so there is a rational curve R passing through x satisfying the condition that $c_1(T_x) \cdot R \leq \dim(X) + 1$.