DUE DATE: JAN. 17, 2008

1. Let R be the set $R = \{x \in \mathbb{R} \mid x > 0\}$ of positive real numbers. In class it was claimed that the operations

$$\begin{aligned} a \oplus b &= a \cdot b, \text{ and} \\ a \odot b &= e^{\ln(a)\cdot\ln(b)} \end{aligned}$$

along with the elements "0" $= 1 \in R$ and "1" $= e \in R$ make R into a ring, i.e., something where addition and multiplication obey "all the rules we're used to".

To get a feeling for what this means, verify directly (i.e., using the definitions of $\oplus$ and $\odot$) that the following identities are true for any x, y, and $z \in R$.

(a) $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$, (i.e., $x(y+z) = xy + xz$).

(b) $(x \oplus y) \odot (x \oplus y) = (x \odot x) \oplus (x \odot y) \oplus (x \odot y) \oplus (y \odot y)$ (i.e., $(x+y)^2 = x^2 + 2xy + y^2$).

REMINDER: The "$\cdot$" in the definition of $\oplus$ and $\odot$ means ordinary multiplication of real numbers.

2. Consider the sum $1 + 3 + 5 + 7 + \cdots + 2n - 1$ of odd numbers. Find a formula for this sum in terms of n (this also means proving the formula!).

3. Let $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$ define F_n recursively by $F_n = F_{n-1} + F_{n-2}$. These are the famous Fibonacci numbers; the first few are shown in the table below:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	$\cdots$

(a) Find the roots of $x^2 - x - 1 = 0$.

(b) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Prove that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. (There is more than one way to do this, and one of the ways is more efficient than the other...)

(c) We would like to prove the formula $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$. Verify that it is true for $n = 0$ and $n = 1$.

(d) Prove the formula above for all n. (You will probably need the "complete induction" version of induction, since in the inductive step you'll want the formula to be true for more than one value of n).

4. If f and g are differentiable functions, the product rule tells us that $\frac{d}{dx} f \cdot g = f' \cdot g + f \cdot g'$. If f and g are functions which are infinitely differentiable (i.e., we can take as many derivatives as we like), prove the product rule for taking higher derivatives:

$$\frac{d^n}{dx^n} f \cdot g = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(n-k)}.$$

Here $f^{(k)}$ means the k-th derivative of f. The binomial identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, valid for $1 \leq k \leq n$ may be useful.