1.

(a) Suppose that $g(z)$ is holomorphic at z_0, and that $g(z_0) \neq 0$. Let n be a positive integer. We want to compute $\text{Res}\left(\frac{g(z)}{(z-z_0)^n};\, z_0\right)$ in terms of g. In order to practice some of the ideas of the course, let's do this in three different ways (each of which are short):

- (i) Explain how you can compute the residue by integration in a small circle around z_0, and then use this integral and Cauchy's integral formula for derivatives to compute the residue.
- (ii) Expand g in a Taylor series around z_0, divide by $(z-z_0)^n$ and find the residue. (To get the final answer you will have to use the formula for the coefficients of the Taylor series in terms of the derivative).
- (iii) Find the residue using short-cut (5).

(b) Does the formula in (a) still hold if g has a zero of order k at z_0 (with $k \geq 1$)?

NOTE: Part (a) is asking for a generalization of short-cut (4), which is the case $n = 2$.

2. Compute P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{10}{z(z-i)(z-2-i)}\, dz$ using both the upper and lower half-plane versions of Proposition 30.1.

NOTE: So far all the integrals we've done were for real-valued functions, and this implies that the poles must come in conjugate pairs: If z_0 is a pole then so is $\overline{z}_0$. We're also used to seeing some correlation between the residues at the corresponding pairs of points; somehow the formulas "look the same". This example shows that this doesn't have to happen in general, and verifies (if we're allowed to use both versions), that both the upper and lower half plane methods produce the same answer, as they must, by the proposition.

3.

(a) Show that $z^6 + 4z^2 - 1$ has exactly two zeros (counted with multiplicity) in the unit disk $\{z \mid |z| < 1\}$.

(SUGGESTION: consider $g(z) = 4z^2 - 1$.)

(b) How many zeros does $z^6 - 4z^5 + z^2 - 1$ have in the unit disk?

4. The difference between Proposition 29.2 and Proposition 30.1 is that the latter allows simple poles on the domain of integration (the real axis), which we deal with by taking hops around the poles in question. In Proposition 29.1 we found out how to do certain kinds of trigonometric integrals by converting them to integrals on the circle, but part of the hypotheses of that proposition was that the resulting function f have no poles on the unit circle.

The purpose of this question is to combine the hop idea of Proposition 30.1 with the trigonometric integral idea of Proposition 29.1, and to give some practice with the definition of the Cauchy principal value and the geometry of the Log function.

(a) Suppose that $P(x, y)$ is a rational function of x and y, such that

$$f(z) = P\left(\tfrac{1}{2}\left(z + \tfrac{1}{z}\right), \tfrac{1}{2i}\left(z - \tfrac{1}{z}\right)\right) \cdot \tfrac{1}{zi}$$

has at most simple poles on the unit circle.

Use the ideas of the proof of Proposition 30.1 to predict a formula for

$$\text{P.V.} \int_0^{2\pi} P\left(\cos(\theta), \sin(\theta)\right)\, d\theta$$

in terms of the residues of f. You do not have to prove your formula, but if you want you can try and justify it by indicating what the ideas of proof should be.

(b) Apply your formula in part (a) to predict the value of P.V. $\displaystyle\int_0^{2\pi} \frac{1}{\cos(\theta) + i\sin(\theta) + 1}\, d\theta.$

Note that the principal value is needed since the denominator is zero when $\theta = \pi$.

In the rest of the problem we will work out the answer to (b) directly to check the prediction.

(c) Sketch the curve $1 + e^{-i\theta}$, $\theta \in [0, 2\pi]$ and check that it passes through the set $\left\{ z \in \mathbb{R} \,\middle|\, z \leq 0 \right\}$ only when $\theta = \pi$. Note that this means that $\text{Log}(1 + e^{-i\theta})$ is defined for all $\theta \neq \pi$.

(d) To compute the integral in (b), first write $\cos(\theta) + i\sin(\theta)$ as $e^{i\theta}$ and then multiply the resulting fraction by $\frac{e^{-i\theta}}{e^{-i\theta}}$. You should be able to figure out the antiderivative of the resulting function.

(e) Use the antiderivative to compute the answer to the integral in (b), and compare it with your prediction. Note that since the integral is a Cauchy principal value, you will have to take a symmetric limit around $\theta = \pi$ when evaluating it.

This assignment is due on or before **Monday, December 13, 2010** at **4 pm**. Please leave the assignment in my mailbox in Jeff 507.
