Approximations in setting up Riemann Sums

1. The area of a ring.

Last class, the issue came up of how to approximate the area of a circular ring, with inner radius \( r \) and outer radius \( r + \Delta r \).

The exact area is

\[
\pi(r + \Delta r)^2 - \pi r^2 = \pi(r^2 + 2r \Delta r + (\Delta r)^2 - r^2) = 2\pi r \Delta r + \pi(\Delta r)^2.
\]

There were some people who felt that \( 2\pi r \Delta r \) would do as well, even though it isn’t the exact formula for the area (it’s missing the \( \pi(\Delta r)^2 \) term).

In fact, that’s correct – when setting up the Riemann sum you can simply ignore the \( \pi(\Delta r)^2 \) part.

How can using an incorrect formula give you a correct answer? Read on to find out.

2. \( \Delta x \) versus \( (\Delta x)^2 \).

Suppose we’re trying to compute a Riemann sum over the interval \([0, 10]\), adding up some formula that involves both \( \Delta x \) and \( (\Delta x)^2 \). (It could be the formula above with \( x \) in place of \( r \) for example.)

Our plan, as always for Riemann sums, is to divide the interval up into \( n \) pieces, evaluate the function on the pieces, and then add up the values on the pieces.

That means we’ll be adding up \( n \) terms. We’ll see a term involving \( \Delta x \) appear \( n \) times in the sum, and a term involving \( (\Delta x)^2 \) appear \( n \) times in the sum. As we increase \( n \),
both $\Delta x$ and $(\Delta x)^2$ decrease, but since we’ll be adding them up $n$ times, it might not be so clear what will happen.

Here’s a little table for various values of $n$, of $n$, $\Delta x$, $(\Delta x)^2$, $n \cdot \Delta z$, and $n \cdot (\Delta x)^2$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Delta x$</th>
<th>$(\Delta x)^2$</th>
<th>$n \cdot \Delta x$</th>
<th>$n \cdot (\Delta x)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>100</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>100</td>
<td>0.1</td>
<td>0.01</td>
<td>10</td>
<td>0.1</td>
</tr>
<tr>
<td>1000</td>
<td>0.01</td>
<td>0.0001</td>
<td>10</td>
<td>0.01</td>
</tr>
<tr>
<td>10000</td>
<td>0.001</td>
<td>0.000001</td>
<td>10</td>
<td>0.001</td>
</tr>
<tr>
<td>10000</td>
<td>0.0001</td>
<td>0.00000001</td>
<td>10</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

(Note: the interval is $[0, 10]$, so $\Delta x = \frac{10}{n}$.)

You can see that as $n$ increases, the contribution from the $n \cdot \Delta x$ term remains constant, but the contribution from the $n \cdot (\Delta x)^2$ term is disappearing. That seems to imply that its contribution to the sum will also disappear as $n$ gets large.

Let’s try this out with a concrete example involving density on a circular shape.

3. Two computations of the mass.

Just to pick a concrete example, let’s suppose that we have a circular shape of radius 10, and that the density of this shape at a distance $r$ from the center is given by

$$\rho(r) = \sqrt{100 - r^2}.$$ 

What is the total mass of the circular shape?

If we divide the circle up into $n$ concentric rings, the mass of each ring is either

$$\sqrt{100 - r^2} \cdot (2\pi r \Delta r + \pi (\Delta r)^2)$$

using the correct expression for the area, or

$$\sqrt{100 - r^2} \cdot 2\pi r \Delta r$$

using the approximate expression for the area.

That means we now have two possible candidates for a sum which approximates the mass:

$$\sum_{i=1}^{n} (\sqrt{100 - r_i^2}) \cdot (2\pi r_i \Delta r + \pi (\Delta r_i)^2) \quad \text{or} \quad \sum_{i=1}^{n} (\sqrt{100 - r_i^2}) \cdot 2\pi r_i \Delta r.$$
Let’s compare the results given by these sums as \( n \) gets large.

\[
\begin{array}{ccc}
\begin{array}{c}
 n \\
\hline
1 \\
10 \\
100 \\
10000 \\
100000 \\
1000000 \\
\end{array}
& \begin{array}{c}
\sum_{i=1}^{n}(\sqrt{100 - r_i^2}) \cdot (2\pi r_i \Delta r + \pi(\Delta r)^2) \\
\sum_{i=1}^{n}(\sqrt{100 - r_i^2}) \cdot 2\pi r_i \Delta r \\
\end{array}
& \begin{array}{c}
3141.592655 \\
2291.178682 \\
2117.320204 \\
2094.640009 \\
2094.419706 \\
2094.397755 \\
\end{array}
\end{array}
\]

As you can see, as \( n \) gets large, the sum without the \((\Delta r)^2\) term is giving the same answer as the sum with the \((\Delta r)^2\) term.

In other words: The contribution of the \((\Delta r)^2\) term makes no difference in the limit, and hence no difference to the final answer.

That’s a powerful trick: When setting up our Riemann sums, we can ignore terms which have \((\Delta r)^2\), or in general, powers of \((\Delta r)\) higher than one.

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*This handout can (soon) be found at*

http://www.mast.queensu.ca/~mikeroth/calculus/calculus.html

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