Fishing limits and the Logistic Equation.[1]

1. The Logistic Equation.

The logistic equation is an equation governing population growth for populations in an environment with a limited amount of resources (for instance, food, or space).

If t is the variable for time, and $P(t)$ the size of the population at time t, then the logistic differential equation is

$$\frac{dP}{dt} = kP(L - P)$$

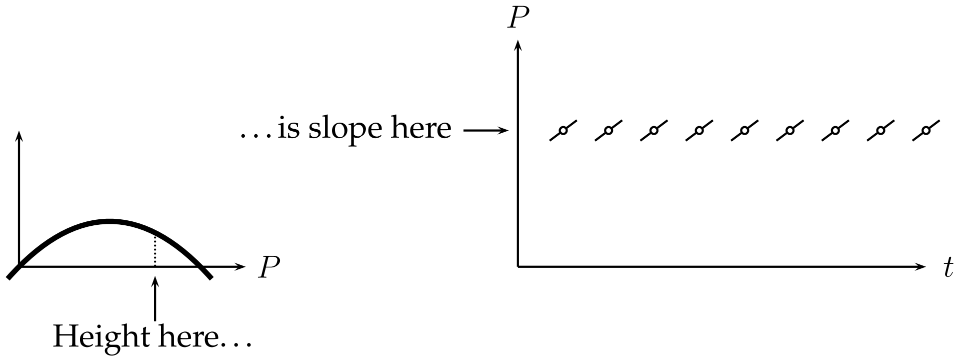
where the constants k and L depend on the particular population and the details of its environment.

In fact, the constant L has a clear physical interpretation, as we'll see below: it is the *carrying capacity* of the environment, the largest population that the environment can support.

One way to try and understand the logistic equation qualitatively is to sketch its *slope field*. To do this, at each point (P, t) we mark a little line, whose slope is obtained by plugging in the coordinates P and t into the equation above.

In the case of the logistic equation, we have a bit of luck: the equation $kP(L - P)$ doesn't depend on the variable t, so the slopes on the slope field will be the same as we move horizontally.

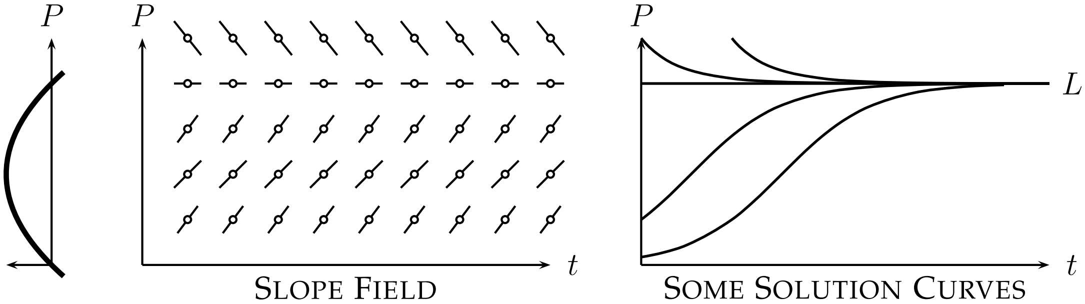
It's worthwhile to look at the equation $kP(L - P)$ a bit more carefully. As a function of P, it is a parabola, with zeros at $P = 0$ and $P = L$. The relationship between the values (the heights) of the parabola and the slopes of the slope field is shown in the diagram below.



In order to analyze the solutions, it is easiest to simply turn the parabola on its side, and draw it next to the graph of the slope field so that the P-axes line up.

[1]This discussion is taken entirely from Arnol'd's book *Ordinary Differential Equations*, p 24–27.

Here is the corresponding sketch of the slope field, and some of the solution curves.



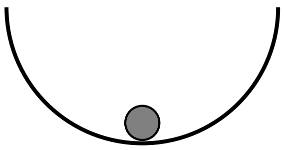
SLOPE FIELD SOME SOLUTION CURVES

From the sketches, we see that if we start with a population P smaller than L, it will climb to approach L. If we start with a population larger than L, P will decrease, again approaching L. This justifies the claim made earlier that L represents the carrying capacity of the environment.

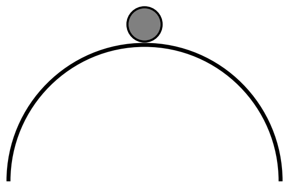
In fact, the observation that all starting possibilities tend to L says even more: it says that $P = L$ is a *stable* equilibrium value.

Roughly, stable means that if we perturb the solution a little (pushing P upwards or downwards a bit) it will tend back to where it started from. In contrast, an *unstable* equilibrium is one where perturbing the solution may cause it to move away from the equilibrium value.

The idea is perhaps best explained informally by the two pictures below.



STABLE



UNSTABLE

Physically we never expect to see an unstable solution. Everything in the world is subject to bumps and fluctuations; the survival of an unstable solution is unrealistic.

2. Two Quota policies.

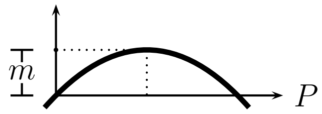
Let's suppose that we are dealing with a population of fish, perhaps confined to a lake, or to a particular breeding environment, and that this population of fish satisfies the logistic differential equation.

We want to allow harvesting of the fish (i.e., fishing), and need to impose some kind of quota to ensure that the fish population continues to survive. We'd like to analyze the effects of two possible quota policies.

The policies are:

> POLICY I (FIXED QUOTA): We fix a number c, and each year we allow c fish to be caught.
>
> POLICY II (PERCENTAGE QUOTA): We fix a number f (for *f*raction of *f*ish) and each year allow $f \cdot P$ fish to be caught.

We need to analyze the effect of these policies on the dynamics of the fish population and decide between them. We also need to decide on the appropriate value of c or f for the policy we choose. Our first concern is for the *sustainability* of the policy – we want to ensure the long term survival of the fish. Our second concern (always making sure that our policy is sustainable) is to have the largest yearly yield of fish.



In the the analysis of each of these two quotas, the maximum value $m = kL^2/4$ (shown in the picture above) achieved by the parabola $kP(L-P)$ will play a part, so it's useful to give it its own name.

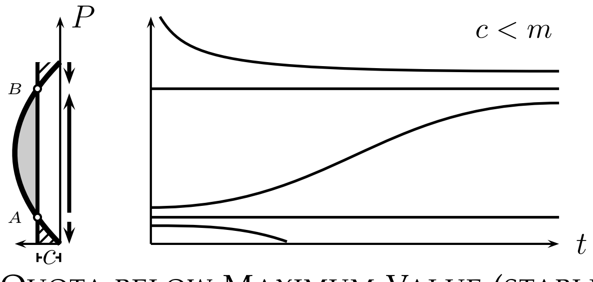
3. Fishing Policy I: Fixed Quota.

Left alone, the population of fish obeys the logistic differential equation. If we allow the removal of c fish per year then the new differential equation satisfied by the population is:

$$\frac{dP}{dt} = kP(L-P) - c.$$

We can analyze this in the same way that we did the logistic equation. The effect of the "$-c$" term is to shift the parabola down. A convenient graphical way to think about this is to draw the line of height c on the parabola, and then think of this line as forming the new P-axis.

Here is a sketch for Policy I where the yearly quota c is less than m.



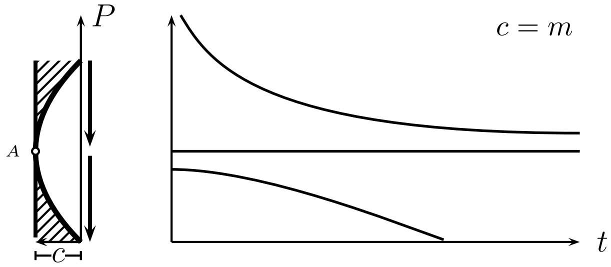
QUOTA BELOW MAXIMUM VALUE (STABLE)

There are two possible equilibrium values, $P = B$ and $P = A$. The value at $P = B$ is stable – small fluctuations of P near the equilibrium head back to B. The equilibrium at $P = A$ is unstable – a small change either sends the fish population shooting up to B (which is good) or downwards to extinction (which is certainly bad).

If we chose Policy I with a value of c like this, we would have a sustainable policy. The level of fish (as long as they started near $P = B$) would remain roughly constant, the reproduction of the fish exactly compensating for the fish removed each year.

In this policy our yearly yield is exactly c (that's the meaning of c). What happens if we try and increase c? As long as $c < m$, we get a picture like the one above, the only difference being the exact location of A and B and the amount of space between them.

Let's look at what happens if we increase c all the way to m.



QUOTA AT MAXIMUM VALUE (UNSTABLE)

Here there is only one equilibrium value, at $P = A$. This equilibrium value is however unstable; if P ever drops below A, then the result is extinction of the population.

Practically, even though it gives the largest yearly yield of fish, $c = m$ is inadmissible as a policy. We expect slight fluctuations in the fish population, and (given the instability of the equilibrium) these would soon lead to extinction. This is not a sustainable solution.

We should also not allow values of c too close to m, there the values of A and B would be close together, and a small fluctuation of the population could send it below A, again resulting in extinction.

Summarizing, it seems that for Policy I, the maximal theoretical yearly limit is m fish per year, but practically we're going to have to insist that $c < m$. How far below m is a combination of our guess at how large random fluctuations in the fish population might be, and our tolerance for risk.

4. Fishing Policy II: Percentage Quota.

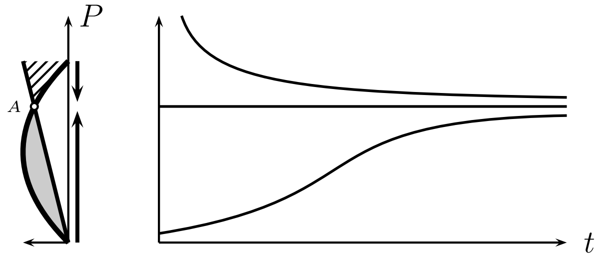
Instead of allowing the removal of a fixed amount c of fish per year, we're going to try fixing a number f, and allowing the removal of fP fish per year. (For instance, $f = 0.20$ would mean that we allow 20% of the fish to be removed each year.)

Under this policy, the differential equation describing the population of fish is

$$\frac{dP}{dt} = kP(L-P) - fP.$$

The effect of the $-fP$ term is again to shift the parabola, although the amount of shifting now depends on the location of P. Let's just do the same thing we did before: draw the line $y = fP$ over the graph of the parabola, and realize that we're looking at the difference in the two equations.

One striking difference between this policy and the previous one is that the equilibrium solution is stable, no matter which fraction f we pick.

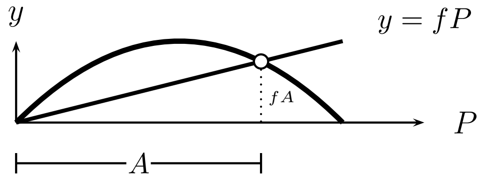


EQUILIBRIUM SOLUTION ALWAYS STABLE

The equilibrium value A is the P coordinate of the intersection of the line and the parabola. Since the equilibrium is stable, this is a sustainable policy.

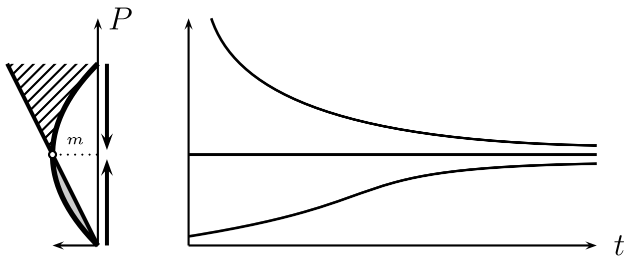
So, once we fix our value of f, the fish population will settle down to A, and each year we'll be getting fA fish.

What we need to do is figure out how to pick f so that fA will be the largest. One slight trouble is that the value of A depends on f. Fortunately there's a way to see the product fA on the graph of the parabola and the line, and that will make it easy to maximize fA.



Since we're looking for fA, and the line is of the form $y = fP$, this is the same as looking for the height of the line when $P = A$. But $P = A$ is exactly the point where the line intersects the parabola (that's how we found A in the first place), so the number of fish we expect to get each year is just the height of the intersection point.

In order to maximize our yearly yield of fish, we just have to make the intersection point the highest possible. And we clearly do that by putting the line through the vertex of the parabola:



LINE WITH MAXIMUM YEARLY YIELD

When we do this, the yearly yield from fishing is exactly m, the height of the vertex of the parabola.

Summarizing, for Policy II, the maximal theoretical yearly limit is m fish per year, and we can do it in a safe, sustainable way. Clearly Policy II (with $f = kL/2$, which puts the line through the vertex) is the best choice: We not only get more fish than with Policy I, but the resulting equilibrium is stable.

5. Concluding remarks.

The fact that Policy II has better stability properties than Policy I is not surprising: Since we're taking a fraction fP of the fish, anytime the population P drops we are effectively lowering our quota as well, giving the fish population time to recover. Under Policy I, we keep removing the same number of fish, potentially devastating the population.

It's somewhat wonderful that the dynamics of these different models of the fish population can be understood by our geometric arguments without ever solving the differential equations.

Many of the modern insights into the nature of differential equations come from emphasizing the geometric properties over explicit solutions; the geometric method is a powerful one.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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