Improper Integrals

1. The infinite in Calculus.

Much of the technical foundation for calculus consists in techniques for dodging problems to do with infinity: When we take the derivative of a function, we take a ratio $(f(x+\Delta x) - f(x))/\Delta x$ as Δx gets "infinitely small", or when integrating, we take a Riemann sum over an interval divided into n parts, and then let n get "infinitely big".

When calculus was first being developed, in the late seventeenth century, people did not know exactly how to describe what this meant, and there was endless controversy and confusion over whether these definitions meant anything at all.

As an example, here is the the philosopher Thomas Hobbs criticizing his archenemy Wallis's attempt to develop calculus:

> "Your scurvy book of *Arithmetica Infinitorum;* where your indivisibles have nothing to do, but as they are supposed to have quantity, that is to say, be *divisibles*."

How to describe what was really going on was resolved with the isolation and use of the idea of *limit*. This re-foundation of calculus and introduction of the rigorous idea of limit didn't happen until the early nineteenth century – almost a century and a half after calculus was first introduced.

The idea of the limit lets us "sneak up" on something we don't understand, and in fact we end up defining the thing we're trying to understand as the limit of all the approximations.

It's a rather subtle trick, and that partly explains why it took so long for the limit to be officially formalized, and why it and the definitions of derivative and integral appear confusing the first time you see them.

2. Improper integrals.

We'd like to expand our tool of integration to include problems where infinity appears in a new way: we'd like to integrate functions over infinitely long intervals, or integrate functions which go upwards to infinity.

Here's an example where we want to integrate over an infinitely long interval:

To be concrete, let's try and integrate the functions $f(x) = 1/\sqrt{(x)}$, $f(x) = 1/x$, and $f(x) = 1/x^2$ "from 1 to ∞", that is, from 1 going on to the right forever.

It's not really clear exactly what this should mean. Let's employ the insight of the limit and try and sneak up on the answer. In each of these cases, let's calculate $\int_1^b f(x)\ dx$ for larger and larger values of b, and see what happens to the answer.

b	$\int_1^b \frac{1}{\sqrt{x}}\ dx$	$\int_1^b \frac{1}{x}\ dx$	$\int_1^b \frac{1}{x^2}\ dx$
10	4.3245553	2.30258509	0.9000000000
100	18.0	4.6051701859	0.9900000000
1000	61.2455532	6.9077552	0.9990000000
10^4	198.0	9.210340371	0.9999000000
10^5	630.4555	11.512925	0.9999900000

As b gets larger and larger, the first two columns also seem to be getting larger and larger, and so it seems like there's no way to make sense of the integral "out to ∞" for those two functions.

In contrast, the third column seems to be getting closer and closer to 1, and we'd expect that the integral from 1 to ∞ of $1/x^2$ might actually make sense, and be equal to 1.

In general, when we want to take an integral out to ∞, say the integral of some function $\int_a^\infty f(x)\ dx$ we do exactly what we did above. We compute the integrals $\int_a^b f(x)\ dx$ for larger and larger values of b. If, as b gets larger, these numbers settle down to a single answer, we call that the answer for the integral out to infinity. If these numbers don't settle down, we say that the answer doesn't exist.

In mathematical symbols, what we've just said is that we're defining

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx.$$

We do something similar when dealing with the second kind of problem – functions whose values go upwards to infinity.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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