Two Wrongs Make a Right
(a.k.a. Separation of variables)

1. When it applies.

Today's topic is to learn about a method, "Separation of Variables" which can be used to solve differential equations of a certain special form.

To use separation of variables, the equation must be of the type

$$\frac{dy}{dx} = \text{(Some function of } y) \cdot \text{(Some function of } x).$$

Here by "y" we really mean a function $y(x)$ of x, but we are using the kind of notation where we don't include those helpful details.

Here are some equations where you can apply separation of variables:

(1)
$$\frac{dy}{dx} = -\frac{x}{y}.$$

This is of the correct form, since we can write

$$-\frac{x}{y} = (\frac{1}{y}) \cdot (-x),$$

which is of the form (a function of y) times (a function of x).

(2)
$$\frac{dy}{dx} = x \cdot y.$$

This is also clearly of the correct form.

The following differential equation is *not* of the right form:

$$\frac{dy}{dx} = y - x.$$

It isn't of the form (a function of y) times (a function of x).

You have to be a little careful when you're deciding however. For instance

(3)
$$\frac{dy}{dx} = xy + x + y + 1.$$

is an equation of the right form, since $xy + x + y + 1 = (y + 1) \cdot (x + 1)$.

2. Separation of Variables.

Starting with a differential equation of the special form

$$\frac{dy}{dx} = \text{(Some function of } y) \cdot \text{(Some function of } x),$$

we do the following steps to solve the differential equation:

$\boxed{1}$ "Separate": Move everything with an x (including the dx) to the right hand side, and everything with a y to the left hand side.

$$\frac{dy}{\text{(The function of } y)} = \text{(The function of } x)\ dx.$$

$\boxed{2}$ "Integrate": Integrate both sides, on the left with respect to y, and on the right with respect to x. DON'T forget the $+C$, which you can put on the side with the x's:

$$\int \frac{dy}{\text{(The function of } y)} = \int \text{(The function of } x)\ dx + C.$$

$\boxed{3}$ "Manipulate": Solve the resulting equation for y as a function of x:

$$y = \ldots?$$

3. Examples.

Let's try this with example (1) from the first page.

Separate:

$$y\ dy = -x\ dx$$

Integrate:

$$\int y\ dy = -\int x\ dx + C, \quad \text{or}$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + C.$$

Manipulate:

$$y^2 = C - x^2, \quad \text{or} \quad y = \sqrt{C - x^2}.$$

Here's example (2) from the first page:

Separate:

$$\frac{dy}{y} = x\ dx$$

Integrate:

$$\int \frac{dy}{y} = \int x\ dx + C, \quad \text{or} \quad \ln|y| = \frac{x^2}{2} + C_1.$$

Manipulate:

$$e^{\ln|y|} = e^{\frac{x^2}{2}+C_1}, \quad \text{or} \quad y = C_2 \cdot e^{\frac{x^2}{2}}.$$

Where $C_2 = e^{C_1}$. It's common to just call the end constant C so that we can write this as

$$y = C \cdot e^{\frac{x^2}{2}}.$$

4. Two Wrongs. . . .

The method of separation of variables works (and is in fact mathematically correct), but two of the steps seem like complete nonsense.

First, in step 1, we start with $\frac{dy}{dx}$ and somehow "split apart the fraction", separating dy and dx.

The problem is, $\frac{dy}{dx}$ *isn't* a fraction – the symbol $\frac{dy}{dx}$ means a *function*.

For instance, if $y(x) = e^{x^2} - \sin(x)$, then $\frac{dy}{dx}$ means

$$\frac{dy}{dx} = 2xe^{x^2} - \cos(x),$$

and so how can you split this up into "dy" and "dx"?

Second, let's consider step 2. When we're working with an equality (like $A = B$), we know that we can perform operations on one side of the equation as long as we perform the identical operations on the other side.

For instance, starting from $A = B$, we could add 5 to both sides: $A + 5 = B + 5$, or maybe divide by 2: $A/2 = B/2$, or maybe take the logarithm: $\ln(A) = \ln(B)$. The important thing is that we can't do something different to each side and still expect to have an equality. Adding 1 to A and 5 to B and claiming that $A + 1 = B + 5$ is nonsense.

In step 2, we integrate both sides of an equality, but we do it with respect to completely different variables! How can the integral of one side with respect to y be equal to the integral of the other side with respect to x? We're doing completely different things to the different sides of the equation!

5. . . . make a Right.

There is a way to make sense of steps 1 and 2, and to justify the method of separation of variables.

Let's imagine that both x and y are functions of a third variable t (so that there is some formula for x in terms of t, and some formula for y in terms of t). In fact, since we're already imagining that y is a function of x (i.e., there is some formula for y in terms of x), we can just plug the formula for x in terms of t into the formula for y in terms of x to get the expression for y in terms of t. (The assumption that there is such a t isn't so hard to justify.)

The chain rule tells us that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},$$

or, that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

The symbols dx and dy when we do step one really stand for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, so that the first step should really look like:

$$\frac{1}{\text{(The function of } y)} \frac{dy}{dt} = \text{(The function of } x) \ \frac{dx}{dt}.$$

We now integrate both sides, but with respect to the variable t:

$$\int \frac{1}{\text{(The function of } y)} \frac{dy}{dt} \ dt = \int \text{(The function of } x) \ \frac{dx}{dt} \ dt.$$

(Don't forget that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are actually *functions*.)

If we now apply substitution to the two different sides (using $u = x(t)$ on the right, $v = y(t)$ on the left), this becomes simply

$$\int \frac{dy}{\text{(The function of } y)} = \int \text{(The function of } x) \ dx.$$

Where we still interpret both sides as being functions of t.

Once we work out the integrals, we get an equality between functions of t, but, since everything is expressed in terms of x's and y's, we can just forget about the t and solve for y in terms of x.

This expression for y in terms of x solves the differential equation, so we can now just forget about the fact that x and y were functions of t, and in fact even forget about t altogether!

It's essentially the argument above that underlies separation of variables. Fortunately we don't need to consciously think about it, since all the difficulties and confusion of the extra parameter are neatly swept under the rug by merely shuffling around the dx's and dy's.

6. Notation in Calculus.

The notation for Calculus has been very carefully thought out (mostly by Leibniz) so that in some sense "the notation does the work". That is, many somewhat complicated procedures appear, using Leibniz' notation, to be no more than simple manipulation of fractions. This can be convenient – working mathematicians (and students) are very happy to have a formal system to guide calculations without the necessity of understanding or thinking about each single step. On the other hand, since it reduced many mathematical arguments to formal manipulation of symbols, it can lead to confusion about the nature of the calculation, or at worst, rob the calculation of all meaning entirely.

The problem is succinctly described by the great mathematician Arnol'd in the introduction to his book *Ordinary Differential Equations*:

> . . . Another of Leibniz' grand achievements is the broad publicizing of analysis (his first publication is an article in 1684) and the development of its algorithms to complete automatization: he thus discovered a method of teaching how to use analysis (and teaching analysis itself) to people who do not understand it at all – a trend that has to be fought against even today.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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