Summary Notes for Vector Calculus

Differentiability

◇ $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}_0$ if there is a vector $\vec{a}$ such that

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\left| f(\vec{x}) - f(\vec{x}_0) - \vec{a}\cdot(\vec{x}-\vec{x}_0)\right|}{\|\vec{x}-\vec{x}_0\|} = 0$$

If such an $\vec{a}$ exists it must be $(\frac{\partial f}{\partial x_1}(\vec{x}_0), \frac{\partial f}{\partial x_2}(\vec{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\vec{x}_0))$ called $D(f)(\vec{x}_0)$.

◇ $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\vec{x}_0$ if there is a $n \times m$ matrix A such that

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\left\| F(\vec{x}) - F(\vec{x}_0) - A(\vec{x}-\vec{x}_0)\right\|}{\|\vec{x}-\vec{x}_0\|} = 0$$

If such an A exists it must be

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\vec{x}_0) & \cdots & \frac{\partial F_1}{\partial x_m}(\vec{x}_0)) \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(\vec{x}_0) & \cdots & \frac{\partial F_n}{\partial x_m}(\vec{x}_0)) \end{pmatrix}$$

called $D(F)(\vec{x}_0)$.

◇ If all the partials $\frac{\partial F_i}{\partial x_j}$ are continuous on a neighbourhood of $\vec{x}_0$ then F is differentiable at $\vec{x}_0$.

Curves

◇ parametrization: $\vec{c}(t) = (x(t), y(t), z(t))$, for $a \le t \le b$

◇ tangent vector: $\vec{c}\,'(t) = (x'(t), y'(t), z'(t))$

◇ speed: $v = |\vec{c}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

◇ unit tangent: $T(t) = \vec{c}\,'(t)/|\vec{c}\,'(t)|$

◇ arc length $s = \int_a^b |\vec{c}\,'(t)|\,dt$

◇ arc length parametrization: $s(t)$ gives the length of the curve from a to t: $s(t) = \int_a^t |\vec{c}\,'(u)|\,du$, thus $\frac{ds}{dt} = |\vec{c}\,'(t)|$, i.e. $ds = |\vec{c}\,'(t)|\,dt$.

Cylindrical Coordinates

$$\begin{cases} x = r\cos(\vartheta) \\ y = r\sin(\vartheta) \\ z = z \end{cases} \qquad \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \end{cases}$$

$r = \sqrt{x^2+y^2}$

$$\vartheta = \begin{cases} \tan^{-1}(y/x) & x > 0 \\ \tan^{-1}(y/x) + 2\pi & x > 0, y < 0 \\ \tan^{-1}(y/x) + \pi & x < 0 \end{cases}$$

$dV = r\,dr\,d\vartheta\,dz$

Spherical Coordinates

$x = \rho\cos(\vartheta)\sin(\phi)$, $\quad 0 \le \phi \le \pi$

$y = \rho\sin(\vartheta)\sin(\phi)$, $\quad 0 \le \vartheta \le 2\pi$

$z = \rho\cos(\phi)$, $\quad 0 \le \rho$

$\rho = \sqrt{x^2+y^2+z^2}$

$$\vartheta = \begin{cases} \tan^{-1}(y/x) & x > 0 \\ \tan^{-1}(y/x) + 2\pi & x > 0, y < 0 \\ \tan^{-1}(y/x) + \pi & x < 0 \end{cases}$$

$\phi = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)$

$dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\vartheta$

General Change of Coordinates

$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where $D' \ni (u,v) \mapsto (x,y) \in D$

Gradient

◇ del or nabla: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

◇ $\text{grad}(f) = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

◇ ∇f points in the direction of greatest increase of f.

◇ $|\nabla f|$ is the rate of increase in this direction.

◇ $\mathrm{D}_u(f)$ is the rate of change of f in the direction of the unit vector $\vec{u}$: $\mathrm{D}_u(f) = \vec{u}\cdot\nabla f$

◇ ∇f is perpendicular to the level sets of f: if $f(x,y)$ is a scalar field on $\mathbb{R}^2$ then ∇f is perpendicular to the level curves of f; if $f(x,y,z)$ is a scalar field on $\mathbb{R}^3$ then ∇f is perpendicular to the level surfaces of f.

Line Integrals

◇ C is a curve parametrized by $\vec{c}(t) = (x(t), y(t), z(t))$ for $a \le t \le b$, F is a vector field on $\mathbb{R}^3$.

◇ $\int_C F\cdot d\vec{s} = \int_a^b F(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt$ where the curve is traversed from $\vec{c}(a)$to $\vec{c}(b)$.

◇ $-C$ is the same curve as C geometrically, but traversed in the opposite direction.

◇ $\int_{-C} F\cdot d\vec{s} = -\int_C F\cdot d\vec{s}$

◇ $\int_C \nabla f\cdot d\vec{s} = f(\text{end}) - f(\text{beginning}) = f(\vec{c}(b)) - f(\vec{c}(a))$

◇ for a scalar field we set:

$\int_C f(x,y)\,ds = \int_a^b f(\vec{c}(t))\,|\vec{c}\,'(t)|\,dt$
$\int_C f(x,y)\,dx = \int_a^b f(\vec{c}(t))\,x'(t)\,dt$
$\int_C f(x,y)\,dy = \int_a^b f(\vec{c}(t))\,y'(t)\,dt$

Conservative Fields

◇ a vector field F is conservative if the line integral $\int_C F \cdot d\vec{s} = 0$ for any closed loop C, equivalently for any path C, $\int_C F \cdot d\vec{s}$ depends only on the endpoints of C.
◇ ∇f is conservative for any scalar field f.
◇ if F is conservative on an open connected region D,then $F = \nabla f$ where $f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot d\vec{s}$ and the line integral is taken along any path in D joining (a,b) to (x,y).
◇ a region D is simply connected if it has no 'holes'.
◇ if $F = P(x,y)\vec{i} + Q(x,y)\vec{j}$ and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ then F is conservative provided that F is continuous on an open simply connected region D and the first partials of P and Q are continuous on D.

Green's Theorem

◇ $\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$ where C is a simple closed curve in the plane, D is the interior of C, and C is traversed with the interior to the left.

Circulation and Curl

◇ $\text{curl}(F) = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$
$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ for $F = \big(P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\big)$.
◇ if F is a vector field and C is a closed curve, the circulation of F around C is $\int_C F \cdot d\vec{r}$.
◇ if $\vec{d}$ is a unit vector the component of $\text{curl}(F)(x,y,z)$ in the direction of $\vec{d}$ is $\lim_{\text{area}\to 0} \frac{\text{circulation around } C}{\text{area in } C}$ where the limit is taken over the family of curves in the plane perpendicular to $\vec{d}$ which shrink to the point (x,y,z).

Surfaces

◇ a surface S is parametrized by $\vec{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ for $(u,v) \in D$ if $\vec{r}(u,v)$ runs over S as (u,v) runs over D.
◇ the sphere of radius a is parametrized by $\vec{r}(\phi,\vartheta) = \big(a\cos(\vartheta)\sin(\phi),\ a\sin(\vartheta)\sin(\phi),\ a\cos(\phi)\big)$, $0 \le \phi \le \pi$, $0 \le \vartheta \le 2\pi$
◇ $dS = |\vec{T}_u \times \vec{T}_v|\,du\,dv$ gives the element of surface area.
◇ $\text{area}(S) = \iint_S dS = \iint_D |\vec{T}_u \times \vec{T}_v|\,du\,dv$.
◇ for the parametrization of the sphere of radius a given above $|\vec{T}_\phi \times \vec{T}_\vartheta| = a^2 \sin(\phi)$
◇ a parametrization $\vec{r}(u,v)$ of a surface S orients a surface via the normal vector $\vec{T}_u \times \vec{T}_v$
◇ a simple closed curve C which bounds an oriented surface S inherits an orientation from S: an upstanding person will traverse C so that S is to her left.

Flux through a Surface

◇ if F a vector field on a surface S parametrized by $\vec{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ $(u,v) \in D$, the flux of F through S is
$\iint_S F \cdot d\vec{S} = \iint_D F\big(\vec{r}(u,v)\big) \cdot \big(\vec{T}_u \times \vec{T}_v\big)\,du\,dv$

Divergence

◇ if $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field, then $\text{div}(F) = \nabla \cdot F = \left(\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}\right)$.
◇ $\text{div}(F)(x,y,z) = \lim_{\text{vol}\to 0} \frac{\text{the flux of } F \text{ through } S}{\text{volume enclosed by } S}$ where the limit is taken over the family of closed surfaces which contain (x,y,z).

Vector Identities

◇ $\left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{grad}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{div}} \left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\}$
◇ $\text{curl}(\text{grad}(f)) = 0$ for any scalar field f
◇ $\text{div}(\text{curl}(F)) = 0$ for any vector field F
◇ if $\text{curl}(F) = 0$ then $F = \nabla f$ for a scalar field f provided F is defined on all of $\mathbb{R}^3$ and the first partials of F are continuous
◇ if $\text{div}(F) = 0$ then $F = \text{curl}(G)$ for some vector field G provided F is defined on all of $\mathbb{R}^3$ and the first partials of F are continuous.

Stokes's Theorem

◇ $\iint_S \text{curl}(F) \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{s}$ where F is a vector field on S with continuous first partials, S is an oriented surface bounded by a simple closed curve ∂S oriented by S.

Divergence Theorem

◇ $\iint_{\partial V} F \cdot d\vec{S} = \iiint_V \text{div}(F)\,dV$ where F is a vector field on the simple solid region V with continuous first partials, and the boundary surface ∂V is oriented with an outward pointing normal.

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