1. For $\mathbf{F}(x,y) = (\ln(x+5) + y^2, 2xy - y^2)$ we have Curl($\mathbf{F}$) $= 2y - 2y = 0$. If $\mathbf{c}$ is the top half of the unit circle oriented from $(-1,0)$ to $(1,0)$, it has the same endpoints as the line segment $\mathbf{c}'$ joining $(-1,0)$ to $(1,0)$ directly.

By the flexibility theorem for curves, this means that $\int_{\mathbf{c}} \mathbf{F} \cdot ds = \int_{\mathbf{c}'} \mathbf{F} \cdot ds$.

Computing the second integral is easy:

$$\begin{array}{rclcrcl} x(t) & = & t & \quad & (x'(\theta), y'(\theta)) & = & (1,0) \\ y(t) & = & 0 & & \mathbf{F} & = & (\ln(t+5), 0) \end{array}$$

$$\text{for } -1 \leq t \leq 1.$$

The dot product of $\mathbf{F}$ and the velocity vector is $\mathbf{F} \cdot (x', y') = \ln(t+5)$ so the integral is

$$\int_{\mathbf{c}'} \mathbf{F} \cdot ds \;=\; \int_{-1}^{1} \ln(t+5)\, dt = ((t+5)\ln(t+5) - t)_{t=-1}^{t=1} = 6\ln(6) - 4\ln(4) - 2.$$

2. For $\mathbf{F}(x,y,z) = (ye^{z^2} - xe^{xy},\ ye^{xy} + \tan(z^2+z+1),\ x^2)$, we have

$$\text{Div}(\mathbf{F}) = -e^{xy} - xye^{xy} + e^{xy} + xye^{xy} = 0.$$

If S is the top half of the unit sphere (oriented upwards) and S' the unit disk (also oriented upwards) the flexibility theorem for surfaces tells us that $\iint_S \mathbf{F} \cdot dS = \iint_{S'} \mathbf{F} \cdot dS$.

Let's do the second integral, since it seems easier:

$$\begin{array}{rclcrcl} x(r,\theta) & = & r\cos(\theta) & \quad & \mathbf{T}_r & = & (\cos(\theta), \sin(\theta), 0) \\ y(r,\theta) & = & r\sin(\theta) & & \mathbf{T}_\theta & = & (-r\sin(\theta), r\cos(\theta), 0) \\ z(r,\theta) & = & 0 & & \mathbf{N} & = & (0,0,r) \end{array}$$

$$\text{for } 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi.$$

The dot product of the vector field with the normal vector is $\mathbf{F} \cdot \mathbf{N} = r^3\cos^2(\theta)$, so the flux integral is

$$\iint_{S'} \mathbf{F} \cdot dS \;=\; \int_0^1 \int_0^{2\pi} r^3 \cos^2(\theta)\, d\theta\, dr = \frac{\pi}{4}$$

3. If we start with $\mathbf{F}(x,y) = (x^2 - y,\ xy - \arcsin(y) + e^{y^3})$ then $\text{Curl}(\mathbf{F}) = y + 1$.

If $\mathbf{c}_1$ is the top half of the unit circle, oriented from $(-1,0)$ to $(1,0)$, and $\mathbf{c}_2$ is the line segment joining $(-1,0)$ to $(1,0)$ and R the region between $\mathbf{c}_1$ and $\mathbf{c}_2$ then Green's theorem tells us that

$$-\int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \int_{\mathbf{c}_2} \mathbf{F} \cdot ds = \iint_R \text{Curl}_2(\mathbf{F})\, dA.$$

Let's use this to compute $\int_{\mathbf{c}_1} \mathbf{F} \cdot ds$ by computing the other two parts of the equality.

We already parametrized the line segment $\mathbf{c}_2$ in question 1:

$$\begin{aligned} x(t) &= t \qquad & (x'(\theta), y'(\theta)) &= (1,0) \\ y(t) &= 0 \qquad & \mathbf{F} &= (t^2, 1) \end{aligned}$$

$$\text{for } -1 \leq t \leq 1.$$

The dot product of the vector field and the velocity vector is $\mathbf{F} \cdot (x', y') = t^2$, and so

$$\int_{\mathbf{c}_2} \mathbf{F} \cdot ds = \int_{-1}^{1} t^2\, dt = \frac{2}{3}.$$

In polar coordinates, the integral of $\text{Curl}(\mathbf{F})$ over R is

$$\iint_R \text{Curl}(\mathbf{F})\, dA = \int_0^1 \int_0^\pi r^2 \sin(\theta) + r\, d\theta\, dr = \frac{2}{3} + \frac{\pi}{2}.$$

Using these values in the equation above, we get

$$-\int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \left(\frac{2}{3}\right) = \frac{2}{3} + \frac{\pi}{2},$$

Or $\displaystyle\int_{\mathbf{c}_1} \mathbf{F} \cdot ds = -\frac{\pi}{2}$.

4. $\mathbf{E}(x,y,z) = (x^2 - y^2,\ e^y - 2z,\ z^2 + 3\sin(y))$ cannot be an electric field (in a static situation), since $\text{Curl}(\mathbf{E}) = (3\cos(y) + 2, 0, 2y) \neq (0,0,0)$.

Similarly, $\mathbf{B}(x,y,z) = (\sin(x), \sin(y), z^2)$ cannot be a magnetic field (in a static or non-static) situation, since $\text{Div}(\mathbf{B}) = \cos(x) + \cos(y) + 2z \neq 0$.