1.

(a) The position of the point on the curve at time t is given by $p(t) = (e^t \cos(t),\ e^t \sin(t))$, which has velocity vector $\vec{v}(t) = (e^t(\cos(t) - \sin(t)),\ e^t(\sin(t) + \cos(t)))$. Their dot product is

$$p(t) \cdot \vec{v}(t) = e^{2t}(\cos^t(t) - \cos(t)\sin(t) + \cos(t)\sin(t) + \sin^2(t)) = e^{2t}.$$

The vectors have length

$$\begin{aligned} \|p(t)\| &= e^t\sqrt{\cos^2(t) + \sin^2(t)} = e^t \\ \|\vec{v}(t)\| &= e^t\sqrt{\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)} = \sqrt{2}e^t \end{aligned}$$

So that by the cross product formula, the cosine of the angle $\theta(t)$ between them, as a function of t is

$$\cos(\theta(t)) = \frac{p(t) \cdot \vec{v}(t)}{\|p(t)\|\,\|\vec{v}(t)\|} = \frac{e^{2t}}{e^t\,\sqrt{2}e^t} = \frac{1}{\sqrt{2}}.$$

Since this is constant, the angle $\theta(t)$ is constant too. In fact from the cosine we see that the constant angle is $\theta = \pi/4$.

(b) To find the arclength, we integrate the speed from $t = 0$ to $t = 5$. In part (a) we calculated that the speed as a function of t is $\|\vec{v}(t)\| = \sqrt{2}e^t$, and so the arclength is

$$\int_0^5 \sqrt{2}\,e^t\ dt\ =\ \sqrt{2}\,e^t\Big|_{t=0}^{t=5}\ =\ \sqrt{2}(e^5 - 1).$$

(c) (Bonus question) If our parameterization is of the form $p(t) = (r(t)\cos(t),\ r(t)\sin(t))$ then the velocity vector at any point is

$$\vec{v}(t) = (r'(t)\cos(t) - r(t)\sin(t),\ r'(t)\sin(t) + r(t)\cos(t))$$

with length

$$\|\vec{v}(t)\| = \sqrt{(r'(t))^2 + (r(t))^2}$$

The dot product of the position vector and the velocity vector is

$$p(t) \cdot \vec{v}(t) = r'(t)r(t).$$

The dot product formula now tells us that the cosine of the angle $\theta(t)$ between $p(t)$ and $\vec{v}(t)$ at time t is

$$\cos\theta(t) = \frac{p(t) \cdot \vec{v}(t)}{\|p(t)\| \, \|\vec{v}(t)\|} = \frac{r'(t)r(t)}{r(t)\sqrt{(r'(t))^2 + (r(t))^2}} = \frac{r'(t)}{\sqrt{(r'(t))^2 + (r(t))^2}}$$

We're supposing that the angle $\theta(t)$ is constant, say equal to θ. If we let $c = \cos(\theta)$, then the equation we need to satisfy is

$$\frac{r'(t)}{\sqrt{(r'(t))^2 + (r(t))^2}} = c,$$

or squaring, that

$$\frac{(r'(t))^2}{(r'(t))^2 + (r(t))^2} = c^2$$

Solving for $r'(t)$, we get the differential equation

$$r'(t) = \sqrt{\frac{c^2}{1 - c^2}}\; r(t)$$

which has solution $r(t) = e^{kt}$ with $k = \sqrt{\frac{c^2}{1-c^2}}$. Remembering that $c = \cos(\theta)$, we compute that $k = \operatorname{cotan}(\theta)$.

This matches the answer from part (a), since for $\theta = \pi/4$, $\operatorname{cotan}(\theta) = 1$.

2. *Velocity and Acceleration*

(a) If $\vec{u}(t) = (u_1(t),\, u_2(t))$, then $\|\vec{u}(t)\|^2 = u_1(t)^2 + u_2(t)^2$, so

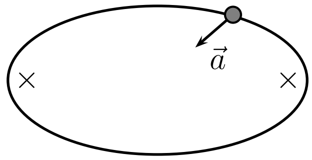
$$\frac{d}{dt}\|\vec{u}(t)\|^2 = 2u_1(t)u_1'(t) + 2u_2(t)u_2'(t) = 2\,\vec{u}(t) \cdot \vec{u}'(t)$$

(b) Suppose that $\vec{v}(t)$ is the velocity vector. Then the derivative $\vec{v}'(t)$ is the acceleration vector $\vec{a}(t)$. The formula from part (a) gives us that

$$\frac{d}{dt}\|\vec{v}(t)\|^2 = 2\,\vec{v}(t) \cdot \vec{a}(t).$$

But the speed is constant if and only if $\|\vec{v}(t)\|^2$ is constant, and that is true if and only if the derivative of $\|\vec{v}(t)\|^2$ with respect to t is zero. By the above formula, that happens if and only if $\vec{v}(t) \cdot \vec{a}(t) = 0$, i.e., if and only if the velocity and acceleration vectors are perpendicular.

(c) The parameterization $x(t) = 2\cos(t)$ and $y = \sin(t)$ does lie on the ellipse $\frac{x^2}{4}+y^2 = 1$, but it does not describe the motion of an object in orbit around a heavier body.



For instance, if we calculate the acceleration vector at time t:

$$\vec{a}(t) = (-2\cos(t),\, -\sin(t))$$

we see that it always points towards the center of the ellipse, while under gravitational acceleration it should always point to one of the two foci (shown with $\times$'s in the picture).

There are many other reasons why this can't be the parameterization of an object in orbit. Among the other possibilities are

- The length of the acceleration vector is not proportional to one over the square of the distance between the object and one of the foci.
- The speed is symmetric with respect to the natural reflectional symmetries of the ellipse. An object in orbit should move faster near the object it is orbiting around, i.e., near one of the foci but not the other.
- The parameterization doesn't sweep out equal areas in equal time.

3. Our parameterization is given by $p(t) = (3t^2,\, t\sin(t),\, e^{2t})$.

(a) When $t = \pi$, $p = p(\pi) = (3\pi^2,\, 0,\, e^{2\pi})$. The velocity vector at time t is

$$\vec{v}(t) = (6t,\, \sin(t) + t\cos(t),\, 2e^{2t}),$$

so the velocity when $t = \pi$ is $\vec{v} = \vec{v}(\pi) = (6\pi,\, -\pi,\, 2e^{2\pi})$.

(b) For $f(x, y, z) = xy + z^2$, the gradient is $\nabla f(x, y, z) = (y, x, 2z)$, so at the point $(3\pi^2,\, 0,\, e^{2\pi})$ we have $\nabla f(3\pi^2,\, 0,\, e^{2\pi}) = (0, 3\pi^2,\, 2e^{2\pi})$.

If we go in the direction given by $\vec{v}$, the instantaneous rate of change of f is

$$\nabla f(p) \cdot \vec{v} = (0, 3\pi^2, 2e^{2\pi}) \cdot (6\pi, -\pi, 2e^{2\pi}) = 4e^{4\pi} - 3\pi^3$$

(c) $f(x(t),\, y(t),\, z(t)) = 3t^3\sin(t) + e^{4t}$, with derivative $\frac{d}{dt}f(x(t), y(t), z(t)) = 9t^2\sin(t) + 3t^3\cos(t) + 4e^{4t}$. When $t = \pi$ this is $9\pi^2 \cdot 0 + 9\pi^3(-1) + 4e^{4\pi} = 4e^{4\pi} - 3\pi^3$.

(d) By the chain rule,

$$\frac{d}{dt} f(p(t)) = \mathbf{D}f(p(t))\,\mathbf{D}p(t).$$

The definition of the gradient is that it is the derivative matrix of f: $\nabla f = \mathbf{D}f$, thought of as a vector. The derivative of the parameterization $p : \mathbb{R} \longrightarrow \mathbb{R}^3$ is the velocity vector $\vec{v}(t)$.

So, the chain rule guarantees that these two computations give the same answer.

4. For the function $f(x, y) = x^2 + \frac{1}{2}y^2$,

(a) The gradient is $\nabla f(x, y) = (2x, y)$.

(b) At the point $(x(t),\, y(t))$, the gradient is $\nabla f(x(t),\, y(t)) = (2x(t),\, y(t))$.

(c) What we want to do is to walk against the gradient, and with the same speed given by the length of the gradient. In other words, if we're at the point $(x(t), y(t))$, we want our velocity vector to be $-\nabla f(x(t), y(t)) = (-2x(t),\, y(t))$. Since the velocity vector is $(x'(t), y'(t))$, that means we want

$$\begin{aligned} x'(t) &= -2\,x(t) \\ y'(t) &= -y(t). \end{aligned}$$

(d) Solving the above differential equations with the initial conditions $x(0) = 1$ and $y(0) = 1$, we get the solutions $x(t) = e^{-2t}$ and $y(t) = e^{-t}$.

(e) The solutions satisfy the conditions $y(t)^2 = x(t)$, so the path lies on the curve $y^2 = x$ which is a sideways parabola. The values of $x(t)$ and $y(t)$ are always positive, so this traces out the part of the parabola which lies in the positive xy quadrant.

5. The only test we know for vector fields to check that they're not conservative is to check the partial derivatives, and see if they match. This is the same as checking if the curl is zero. If the curl *is* zero, and if $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, then we know that there is a function f with $\nabla f = \mathbf{F}$.

(a) $\mathrm{curl}(\mathbf{F}) = (2z, 0, 0)$, so $\mathbf{F}$ is not conservative.

(b) $\mathrm{curl}(\mathbf{F}) = (0, 0, 6x^2 - 2y)$, so this $\mathbf{F}$ is not conservative either.

(c) $\mathrm{curl}(\mathbf{F}) = (0, 0, 0)$, so since $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, it must be the gradient of some function. One of the possibilities is $f(x, y, z) = x\cos(y) - z\sin(x)$.

(d) $\mathrm{curl}(\mathbf{F}) = (0, \cos(z) - \sin(z), 0)$, so this also isn't a conservative vector field.