DUE DATE: OCT. 20, 2004

1. Let f be a function, and $\mathbf{F}$ and $\mathbf{G}$ vector fields on $\mathbb{R}^3$. State whether each of the following expressions is a function, a vector field, or meaningless.

(a) $\text{grad}(\text{grad}(f))$ (b) $\text{Curl}(\text{grad}(f)) - \mathbf{F}$ (c) $\text{Curl}(\text{Curl}(\mathbf{F})) - \mathbf{G}$

(d) $\text{Curl}(\mathbf{F}) \cdot \mathbf{G}$ (e) $\text{Div}(\text{Div}(\mathbf{F}))$ (f) $\text{Div}(\text{Curl}(\text{grad}(f)))$

2. Compute Div and Curl for the following vector fields:

(a) $\mathbf{F}(x, y, z) = (x,\ y,\ z)$ (b) $\mathbf{F}(x, y, z) = (yz,\ xz,\ xy)$
(c) $\mathbf{F}(x, y, z) = (3x^2y,\ x^3 + y^3,\ z^4)$ (d) $\mathbf{F}(x, y, z) = (e^x \cos(y) + z^2,\ e^x \sin(y) + xz,\ xy)$

3.

(a) Is there a vector field $\mathbf{F}$ such that $\text{Curl}(\mathbf{F}) = (xy^2,\ yz^2,\ zx^2)$? Explain.

(b) Is there a vector field $\mathbf{F}$ so that $\text{Curl}(\mathbf{F}) = (2, 1, 3)$? If so, find one.

4. A vector field $\mathbf{F}$ is called *incompressible* if $\text{Div}(\mathbf{F}) = 0$, and *irrotational* if $\text{Curl}(\mathbf{F}) = 0$. A function f is called *harmonic* if $\Delta f = 0$. (NOTE: Δ isn't the gradient, it's the Laplacian.)

(a) Show that any vector field of the form $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$ is irrotational.

(b) Show that any vector field of the form $\mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y))$ is incompressible.

(c) Find constants a, b, and c so that the vector field $\mathbf{F}(x, y, z) = (3x - y + az,\ bx - z,\ 4x + cy)$ is irrotational. For these values of a, b, and c, find the function f with $\nabla f = \mathbf{F}$.

(d) If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ which is both incompressible and irrotational, show that $\mathbf{F}$ is the gradient of a harmonic function f.

5. Decompose each of these vector fields $\mathbf{F}$ as the sum of an irrotational and an incompressible vector field. Is such a decomposition unique?

(a) $\mathbf{F}(x, y, z) = (x^2 + e^{yz},\ x^2z^2,\ \sin(z))$ (b) $\mathbf{F}(x, y, z) = (x + y + z,\ y^2 + 1,\ \ln(xyz))$.