REDUCTION RULES FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

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ABSTRACT. Let G be a semisimple algebraic group over an algebraically-closed field of characteristic zero. In this note we show that every regular face of the Littlewood-Richardson cone of G gives rise to a *reduction rule*: a rule which, given a problem "on that face" of computing the multiplicity of an irreducible component in a tensor product, reduces it to a similar problem on a group $\overline{\mathrm{G}}$ of smaller rank.

In the type A case this result has already been proved by Derksen and Weyman using quivers, and by King, Tollu, and Toumazet using puzzles. The proof here is geometric and type-independent.

Keywords: Homogeneous variety, Littlewood-Richardson coefficient, Littlewood-Richardson cone.

CONTENTS

1. INTRODUCTION

This note is concerned with *reduction rules* — rules reducing the problem of computing the multiplicity of an irreducible component in a tensor product of G-representations to a similar problem on a group $\overline{\mathrm{G}}$ of smaller rank. The main result is that every regular codimension-r face of the Littlewood-Richardson cone of G gives rise to a rule reducing every problem on that face to a group whose rank is r less than the rank of G.

Let G be a semisimple algebraic group over an algebraically closed field of characteristic zero. For a dominant weight μ and a representation V of G we denote by $\mathrm{mult}_{\mathrm{G}}(\mathrm{V}_\mu, \mathrm{V})$ the multiplicity of the irreducible G-representation V_μ in V.

For any $k \geqslant 2$ the *Littlewood-Richardson cone* $\mathcal{C}(k)$ is defined as the rational cone generated by $(\mu_1, \ldots, \mu_k, \mu)$ such that V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. It is known that $\mathcal{C}(k)$

2010 *Mathematics Subject Classification*. Primary 14L35; Secondary 17B10.
Research partially supported by an NSERC grant.

is polyhedral, and minimal equations equations for $\mathcal{C}(k)$ are known through the work of Belkale-Kumar [BK] and Ressayre [R]. A face of $\mathcal{C}(k)$ is called *regular* if it intersects the locus of strictly dominant weights.

By the results in [R], the regular faces of $\mathcal{C}(k)$ are described by the data of a subset I of the simple roots and elements $w_1,\ldots, w_k$, and w of the Weyl group of G satisfying some conditions relative to I (see (2.6.1) for the exact conditions). A point $(\mu_1,\ldots,\mu_k,\mu)\in\mathcal{C}(k)$ is on the face described by this data if and only if the weight $\sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu$ can be written as a $\mathbf{Q}$-linear combination of elements in I.

Suppose that this last condition holds. Then let $\overline{\mathrm{G}}$ be the semisimple part of the parabolic subgroup P_{I} determined by I and $\overline{\mu}_1,\ldots, \overline{\mu}_k$, and $\overline{\mu}$ be the restriction of the weights $w_1^{-1}\mu_1$, $\ldots$, $w_k^{-1}\mu_k$ and $w^{-1}\mu$ respectively to $\overline{\mathrm{G}}$ (see §2.3 and the examples in §4 for a more precise description of this process). The main result of this paper is the construction of a geometric map $(\overline{\mathrm{G}}/\overline{\mathrm{B}})^{k+1} \longrightarrow (\mathrm{G}/\mathrm{B})^{k+1}$ such that pullback of global sections of a particular line bundle induces an isomorphism of vector spaces

$$(\mathrm{V}_{\mu_1}\otimes\cdots\otimes \mathrm{V}_{\mu_k}\otimes \mathrm{V}_{\mu}^{*})^{\mathrm{G}} \xrightarrow{\ \sim\ } (\mathrm{V}_{\overline{\mu}_1}\otimes\cdots\otimes \mathrm{V}_{\overline{\mu}_k}\otimes \mathrm{V}_{\overline{\mu}}^{*})^{\overline{\mathrm{G}}}.$$

Taking dimensions then gives the equality

$$\mathrm{mult}_{\mathrm{G}}(\mathrm{V}_{\mu},\mathrm{V}_{\mu_1}\otimes\cdots\otimes \mathrm{V}_{\mu_k}) = \mathrm{mult}_{\overline{\mathrm{G}}}(\mathrm{V}_{\overline{\mu}},\mathrm{V}_{\overline{\mu}_1}\otimes\cdots\otimes \mathrm{V}_{\overline{\mu}_k}),$$

yielding a reduction rule.

One might guess that a reduction rule occurs because the individual weights $\mu_1,\ldots, \mu_k$, and μ are somehow themselves "special", e.g., somehow come from a group of smaller rank. However, since the faces in question are regular, at a general point of each face all the weights are strictly dominant, and so in some sense generic. It is instead the special configuration of the multiplicity problem — as witnessed by the location of the point $(\mu_1,\ldots,\mu_k,\mu)$ on the boundary of $\mathcal{C}(k)$ — that allows the reduction.

Given the data of I and $w_1,\ldots, w_k$, and w, it is easy to write out explicitly what the corresponding reduction rule does, and examples are given in §4.

An elementary way to describe $\overline{\mathrm{G}}$ is to note that its Dynkin diagram is the full subdiagram of the Dynkin diagram for G corresponding to the simple roots in I. If the resulting subdiagram is disconnected then $\overline{\mathrm{G}}$ is a product of simple groups and hence the reduction rule can also be interpreted as a factorization rule. Under this name, the main result of this note was already known in the type A case and was proved independently by Derksen and Weyman [DW, Theorem 7.14] using quivers and by King, Tollu, and Toumazet [KTT, Theorem 1.4] using puzzles. The proof here is geometric and type-independent.

In type A the Littlewood-Richardson coefficients are also the structure constants in the cohomology rings of the Grassmanians G/P for maximal parabolic subgroups P, and one might hope to generalize the reduction rules for type A in this direction instead. For results along this line, see the forthcoming paper [KP] of Kevin Purbhoo and Allen Knutson.

Acknowledgements. The idea that such reduction rules should hold occurred in joint work ([DR1] and [DR2]) with my colleague Ivan Dimitrov, and several of the ideas used in the proof of the main theorem were developed in [DR1]. I am also greatful to Ivan for valuable discussions on some aspects of the present paper. I thank Kevin Purbhoo for telling me about the paper of Derksen and Weyman and his work with Allen Knutson, as well as for advice on the examples. Explicit instances of the examples were computed with the help of the computer program LiE [vLCL].

2. Preliminary material

2.1. Notation and conventions. Throughout this note we fix a semisimple connected algebraic group G, a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ and a maximal torus $\mathrm{T} \subset \mathrm{B}$. Related groups, whose definition depends on the choice of a subset I of simple roots, are discussed in §2.3. The Lie algebras of algebraic groups are denoted by fraktur letters, e.g. $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, etc. We use the term "weight" both for characters of T and weights of $\mathfrak{t}$. For a dominant weight μ we denote by V_μ the irreducible G-representation of highest weight μ.

Let Δ denote the set of roots of G (with respect to T). For any subset $\Phi \subset \Delta$ we denote by $\mathrm{span}_{\mathbf{Z}} \Phi$ the set of integer combinations of elements of Φ. Similarly, $\mathrm{span}_{\mathbf{Q}_{\geqslant 0}} \Phi$ and $\mathrm{span}_{\mathbf{Z}_{\leqslant 0}} \Phi$ denote respectively the set of non-negative rational combinations and non-positive integer combinations of elements of Φ.

We denote the Weyl group of G by $\mathcal{W}$ and use $\ell(w)$ for the length of any $w \in \mathcal{W}$. We are working over an algebraically closed field of characteristic zero; for notational convenience we will assume that the field is $\mathbf{C}$.

2.2. Inversion sets. Let Δ^+ be the set of positive roots of $\mathfrak{g}$ (with respect to B). Following Kostant [K, Definition 5.10], for any element w of the Weyl group $\mathcal{W}$ we define Φ_w, the *inversion set* of w, to be the set of positive roots sent to negative roots by w, i.e.,

$$\Phi_w := w^{-1}\Delta^- \cap \Delta^+.$$

For a subset Φ of Δ^+, we set $\Phi^c := \Delta^+ \setminus \Phi$. From the definition it follows easily that $\Phi_{w_0 w} = \Phi_w^c$ and that $w^{-1}\Delta^+ = \Phi_w^c \sqcup -\Phi_w$, and we will use these formulas without comment in the rest of the note.

2.3. Discussion of G_I and $\overline{\mathrm{G}}$. Given a subset I of simple roots, let P_I be the corresponding parabolic subgroup, G_I the reductive part (i.e., the Levi component) of P_I, and $\overline{\mathrm{G}}$ the semi-simple part of P_I. We define Δ_I to be the roots of G_I. Equivalently Δ_I is the subset of Δ consisting of those roots in $\mathrm{span}_{\mathbf{Z}} \mathrm{I}$. We denote by Δ_I^+ the intersection $\Delta_\mathrm{I} \cap \Delta^+$, i.e., the positive roots of G_I. Equivalently Δ_I^+ is the subset of Δ consisting of those roots in $\mathrm{span}_{\mathbf{Z}_{\geqslant 0}} \mathrm{I}$. As remarked in the introduction, the Lie algebra $\overline{\mathfrak{g}}$ has an elementary description: the Dynkin diagram of $\overline{\mathfrak{g}}$ is the complete subdiagram of the Dynkin diagram of $\mathfrak{g}$ containing the nodes corresponding to the simple roots in I.

By definition, $\mathrm{T} \subseteq \mathrm{G}_\mathrm{I}$. Let A be the connected component of the center of G_I. Then $\mathrm{A} \subseteq \mathrm{T}$ and $\mathrm{A} \cap \overline{\mathrm{G}}$ is a finite group. The natural map $\overline{\mathrm{G}} \times \mathrm{A} \longrightarrow \mathrm{G}_\mathrm{I}$ sending a pair of elements to their product is a surjective map with finite kernel and thus induces an isomorphism at the level of Lie algebras. We will need to use a specific fact about the resulting direct sum decomposition of $\mathfrak{g}_{\mathrm{I}}$ and so we describe this decomposition in more detail below.

Let $\overline{\mathrm{T}}$ be the connected component of $\mathrm{T} \cap \overline{\mathrm{G}}$, so that $\overline{\mathrm{T}}$ is a maximal torus for $\overline{\mathrm{G}}$, and let $\bar{\mathfrak{t}} = \mathrm{Lie}(\overline{\mathrm{T}})$. Since I is a set of simple roots for $\overline{\mathrm{G}}$, the restriction of the roots in I to $\bar{\mathfrak{t}}$ is a basis (over **C**) of the dual of $\bar{\mathfrak{t}}$. Hence, letting $\mathfrak{a} \subseteq \mathfrak{t}$ be the subalgebra annihilated by the roots in I we obtain a direct sum decomposition $\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathfrak{a}$.

By the definition of $\mathfrak{a}$ we have the following result which we record for later use:

Lemma (2.3.1) — If $\gamma \in \mathrm{span}_{\mathbf{Q}}\,\mathrm{I}$, then the restriction of γ to $\mathfrak{a}$ is zero.

In particular, for any root α of $\bar{\mathfrak{g}}$, any $x \in \bar{\mathfrak{g}}^{\alpha}$ and $a \in \mathfrak{a}$ we have $[a,x] = \alpha(a)x = 0 \cdot x = 0$ and hence the decomposition of $\mathfrak{t}$ extends to a direct sum decomposition $\mathfrak{g}_{\mathrm{I}} = \bar{\mathfrak{g}} \oplus \mathfrak{a}$.

Setting $\mathrm{B}_{\mathrm{I}} := \mathrm{G}_{\mathrm{I}} \cap \mathrm{B}$ and $\overline{\mathrm{B}} := \overline{\mathrm{G}} \cap \mathrm{B}$ then B_{I} and $\overline{\mathrm{B}}$ are Borel subgroups of G_{I} and $\overline{\mathrm{G}}$ respectively. The direct sum decomposition of $\mathfrak{g}_{\mathrm{I}}$ restricts to a decomposition $\mathfrak{b}_{\mathrm{I}} = \bar{\mathfrak{b}} \oplus \mathfrak{a}$. Note that B_{I} and $\overline{\mathrm{B}}$ have the same unipotent part (equivalently, $\mathfrak{b}_{\mathrm{I}}$ and $\bar{\mathfrak{b}}$ have the same nilpotent part); the difference between the two groups being in their maximal tori.

Restriction of weights. Given a weight μ and an element $w \in \mathcal{W}$ we will use $\overline{\mu}$ and μ' for the restrictions of $w^{-1}\mu$ to $\bar{\mathfrak{t}}$ and $\mathfrak{a}$ respectively under the splitting $\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathfrak{a}$ above. This notation omits the element $w \in \mathcal{W}$ used, but any time we use this notation we will be careful to explicitly specify which element w is meant for that particular restriction.

In the reduction theorem it is implicit that if μ is dominant then the restriction $\overline{\mu}$ will also be dominant. For completeness, let us see why this is true. Let $\kappa(\cdot,\cdot)$ be the Killing form, and suppose that $w \in \mathcal{W}$ is such that $\Phi_w \cap \Delta_{\mathrm{I}}^{+} = \emptyset$; this hypothesis will hold for all w we use when reducing to $\bar{\mathfrak{t}}$. Since $w\Delta_{\mathrm{I}}^{+} \subseteq \Delta^{+}$, if μ is dominant with respect to $\mathfrak{b}$ then $\kappa(w^{-1}\mu, \alpha) = \kappa(\mu, w\alpha) \geqslant 0$ for all $\alpha \in \Delta_{\mathrm{I}}^{+}$ and thus the restriction $\overline{\mu}$ of $w^{-1}\mu$ to $\bar{\mathfrak{t}}$ is dominant with respect to $\bar{\mathfrak{b}}$. Note that this argument also shows that the restriction of a strictly dominant weight is again strictly dominant, and that the restriction of an integral weight is integral with respect to $\overline{\mathrm{T}}$. For this reason we will also refer to the process as "restricting the weight to $\overline{\mathrm{T}}$".

Surjections and $\mathfrak{b}_{\mathrm{I}}$-invariants. We will need the following result giving a condition ensuring that a surjection of $\mathfrak{b}_{\mathrm{I}}$-modules induces an isomorphism of $\mathfrak{b}_{\mathrm{I}}$-invariants.

Lemma (2.3.2) — Suppose that $0 \longrightarrow \mathrm{E}_1 \longrightarrow \mathrm{E}_2 \longrightarrow \mathrm{E}_3 \longrightarrow 0$ is an exact sequence of $\mathfrak{b}_{\mathrm{I}}$-modules, and that no weight of E_1 is contained in $\Delta_{\mathrm{I}}^{+} \cup \{0\}$. Then the induced map $\mathrm{E}_2^{\mathfrak{b}_{\mathrm{I}}} \longrightarrow \mathrm{E}_3^{\mathfrak{b}_{\mathrm{I}}}$ of $\mathfrak{b}_{\mathrm{I}}$-invariants is an isomorphism.

Proof. The first four terms of the long exact sequence arising from taking $\mathfrak{b}_{\mathrm{I}}$-invariants is

$$0 \longrightarrow \mathrm{E}_1^{\mathfrak{b}_{\mathrm{I}}} \longrightarrow \mathrm{E}_2^{\mathfrak{b}_{\mathrm{I}}} \longrightarrow \mathrm{E}_3^{\mathfrak{b}_{\mathrm{I}}} \longrightarrow \mathrm{H}^1(\mathfrak{b}_{\mathrm{I}}, \mathrm{E}_1).$$

By hypothesis, the zero weight does not appear in E_1, and hence E_1 has no $\mathfrak{t}$-invariants, and so no $\mathfrak{b}_{\mathrm{I}}$-invariants, i.e., $\mathrm{E}_1^{\mathfrak{b}_{\mathrm{I}}} = 0$. Let $\mathfrak{b}_{\mathrm{I}}^{+} = [\mathfrak{b}_{\mathrm{I}}, \mathfrak{b}_{\mathrm{I}}]$ be the nilpotent radical of $\mathfrak{b}_{\mathrm{I}}$. Since taking $\mathfrak{t}$ invariants is exact, the Hochschild-Serre spectral sequence for the cohomology of $\mathfrak{b}_{\mathrm{I}}$ degenerates and we have $\mathrm{H}^i(\mathfrak{b}_{\mathrm{I}}, \mathrm{E}_1) = \mathrm{H}^i(\mathfrak{b}_{\mathrm{I}}^{+}, \mathrm{E}_1)^{\mathfrak{t}}$ for all $i \geqslant 0$, and in particular for $i = 1$. The degree one piece of the complex computing $\mathfrak{b}_{\mathrm{I}}^{+}$-cohomology is $\mathrm{C}^1(\mathfrak{b}_{\mathrm{I}}^{+}, \mathrm{E}_1) = (\mathfrak{b}_{\mathrm{I}}^{+})^{*} \otimes \mathrm{E}_1$.

By hypothesis no weight of E_1 lies in Δ_{I}^+, hence $\mathrm{C}^1(\mathfrak{b}_{\mathrm{I}}^+, \mathrm{E}_1)$ has no $\mathfrak{t}$-invariants. Since the differential maps of the complex are $\mathfrak{t}$-equivariant this gives $\mathrm{H}^1(\mathfrak{b}_{\mathrm{I}}^+, \mathrm{E}_1)^{\mathfrak{t}} = 0$. □

2.4. The Borel-Weil theorem. Let $\mathrm{X} := \mathrm{G}/\mathrm{B}$ and let $e \in \mathrm{X}$ be the image of $1_{\mathrm{G}} \in \mathrm{G}$ under the quotient map. The restriction map sending a vector bundle $\mathcal{E}$ on X to its fibre E over $e \in \mathrm{X}$ induces an equivalence of categories between the G-equivariant bundles on X and representations of B. We will use the following special case of that equivalence in establishing the reduction rule:

Principle (2.4.1) — Let $\mathcal{E}$ be a G-equivariant vector bundle on X, and E the fibre over $e \in \mathrm{X}$. then restriction of global sections to the fibre E induces an isomorphism $\mathrm{H}^0(\mathrm{X}, \mathcal{E})^{\mathrm{G}} \xrightarrow{\sim} \mathrm{E}^{\mathrm{B}}$.

For any weight λ we denote by L_λ the G-equivariant line bundle on X corresponding to the one-dimensional B-representation $\mathbf{C}_{-\lambda}$, i.e., the representation where B acts through its quotient T with weight $-\lambda$. The Borel-Weil theorem identifies the G-representation $\mathrm{H}^0(\mathrm{X}, \mathrm{L}_\lambda)$ for any weight λ. The main step in the proof of the Borel-Weil theorem is the following result.

Lemma (2.4.2) — Suppose that L is a G-equivariant line bundle on X, $x \in \mathrm{X}$ any point, and B_x the stabilizer subgroup of X. Using L_x for the fibre of L at x and setting $\mathrm{V} = \mathrm{H}^0(\mathrm{X}, \mathrm{L})$ then the B_x-equivariant restriction map $\mathrm{V} \longrightarrow \mathrm{L}_x$ at x identifies V as the unique irreducible representation of G (if one exists) which has a B_x-equivariant surjection onto the one-dimensional B_x-representation L_x. If no such irreducible representation exists then $\mathrm{V} = 0$.

If λ is dominant then one can show that $\mathrm{H}^0(\mathrm{X}, \mathrm{L}_\lambda) \neq 0$. Since the only surjective B-equivariant quotient map from an irreducible representation V onto a one-dimensional representation is projection is onto the lowest weight vector of V, if $\mathrm{L} = \mathrm{L}_\lambda$ and $x = e$ then Lemma 2.4.2 yields the Borel-Weil theorem:

Theorem (2.4.3) — For any weight λ

$$\mathrm{H}^0(\mathrm{X}, \mathrm{L}_\lambda) = \begin{cases} \mathrm{V}_\lambda^* & \text{if } \lambda \text{ is a dominant weight} \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\mathrm{X}_{\mathrm{I}} := \mathrm{G}_{\mathrm{I}}/\mathrm{B}_{\mathrm{I}}^{\mathrm{op}}$. Then X_{I} is isomorphic to $\mathrm{G}_{\mathrm{I}}/\mathrm{B}_{\mathrm{I}}$ as a G_{I}-variety, but the image of $1_{\mathrm{G}_{\mathrm{I}}}$ under the quotient map $\mathrm{G}_{\mathrm{I}} \longrightarrow \mathrm{X}_{\mathrm{I}}$ has stabilizer B^{op} instead of B. The only surjective B^{op}-equivariant quotient map from an irreducible representation V onto a one-dimensional representation is the projection onto the highest weight vector of V. Applying Lemma 2.4.2 and the splitting of $\mathfrak{g}_{\mathrm{I}}$ from §2.3 then gives the following version of the Borel-Weil theorem for X_{I}:

Theorem (2.4.4) — Suppose that L is a G_{I} equivariant line bundle on X_{I} with torus weight ν at the image of $1_{\mathrm{G}_{\mathrm{I}}}$ in X_{I}, and let $\overline{\nu}$ and ν' be the restrictions of ν to $\overline{\mathfrak{t}}$ and $\mathfrak{a}$ respectively under the splitting from §2.3. Then as a $\mathfrak{g}_{\mathrm{I}}$-module

$$\mathrm{H}^0(\mathrm{X}_{\mathrm{I}}, \mathrm{L}) = \begin{cases} \mathrm{V}_{\overline{\nu}} \otimes \mathbf{C}_{\nu'} & \text{if } \overline{\nu} \text{ is a dominant weight for } \overline{\mathfrak{t}} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if L is the restriction to X_I of a globally generated line bundle under some embedding $\varphi\colon X_I \longrightarrow X$, then $H^0(X_I, L) \neq 0$ and hence only the first alternative above applies. This will be the case in the application of Theorem 2.4.4 in Proposition 2.5.2 below.

2.5. Schubert Varieties. For any element $w \in \mathcal{W}$ of the Weyl group the *Schubert variety* X_w is defined by

$$X_w := \overline{B\dot{w}B/B} \subseteq G/B = X$$

where $\dot{w}$ is any lift of w to G. Since everything we define using w will be independent of the lift, we will almost always omit mention of lifting and just use w in place of $\dot{w}$. The one exception to this convention is Proposition 2.5.2 below where we explicitly consider the lift in order to show that the construction in the proposition is independent of the lifting.

Recall that the classes of the Schubert cycles $\{[X_w]\}_{w\in\mathcal{W}}$ give a basis for the cohomology ring $H^*(X, \mathbf{Z})$ of X. Each $[X_w]$ is a cycle of complex dimension $\ell(w)$. The dual Schubert cycles $\{[\Omega_w]\}_{w\in\mathcal{W}}$, given by $\Omega_w := X_{w_0 w}$, also form a basis. Each $[\Omega_w]$ is a cycle of complex codimension $\ell(w)$.

Remark. If w_1,..., w_k, and $w \in \mathcal{W}$ are such that $\ell(w) = \sum \ell(w_i)$, then the intersection $\cap_{i=1}^{k}[\Omega_{w_i}]\cdot[X_w]$ is a number. This number is the coefficient of $[\Omega_w]$ when writing the product $\cap_{i=1}^{k}[\Omega_{w_i}]$ in terms of the basis $\{[\Omega_v]\}_{v\in\mathcal{W}}$.

To reduce notation we also use w to refer to the point $wB/B \in X_w \subseteq X$. In particular for the identity element $e \in \mathcal{W}$, $X_e = \{e\}$. Note that $e \in X$ is also the image of 1_G under the projection from G onto X.

Open affine cells of Schubert varieties. For any $v \in \mathcal{W}$ the variety $U_v := BvB/B \subseteq X_v$ is B-stable open affine subset of X_v containing v and isomorphic to affine space $\mathbf{A}^{\ell(v)}$. Since U_v is B-stable its coordinate ring $H^0(U_v, \mathcal{O}_{U_v})$ decomposes into T-eigenspaces. Explicitly, $U_v = \operatorname{Spec}(\mathbf{C}[z_{-\alpha}]_{\alpha\in\Phi_{v^{-1}}})$ where each $z_{-\alpha}$ is an independent variable on which T acts via the weight $-\alpha$. The origin of this affine space corresponds to the point v.

For a sequence $\underline{v} = (v_1, \ldots, v_k)$ of elements of $\mathcal{W}$ we set $U_{\underline{v}} = U_{v_1} \times \cdots \times U_{v_k}$. For any weight δ let $H^0(U_{\underline{v}}, \mathcal{O}_{U_{\underline{v}}})_\delta$ be the subspace of $H^0(U_{\underline{v}}, \mathcal{O}_{U_{\underline{v}}})$ of T-eigenfunctions where T acts via δ. The above description of U_v immediately gives the following easy result.

Lemma (2.5.1) — For any sequence $\underline{v}$, if $\delta \notin \operatorname{span}_{\mathbf{Z}_{\leqslant 0}} \Delta^+$ then $H^0(U_{\underline{v}}, \mathcal{O}_{U_{\underline{v}}})_\delta = 0$.

We now come to the main constructions of this section.

Proposition (2.5.2) — Let v be an element of the Weyl group such that $\Delta_I^+ \subseteq \Phi_{v^{-1}}$, $\dot{v}$ any lift of v to G, and $\Psi\colon G_I \longrightarrow G$ the map defined by $\Psi(g) = g\dot{v}$ for all $g \in G_I$. Then

(*a*) The image of G_I under the composite map $G_I \xrightarrow{\Psi} G \longrightarrow X$ is isomorphic to $X_I := G_I/B_I^{op}$ and induces a G_I-equivariant embedding $\psi_v\colon X_I \longrightarrow X$, independent of the lift $\dot{v}$ chosen (here G_I acts on X through its inclusion $G_I \hookrightarrow G$ as a subgroup of G).

(*b*) The image of ψ_v lies in X_v. Setting $\mathrm{U}_v = \mathrm{B}\dot{v}\mathrm{B}/\mathrm{B}$ to be the B-stable open affine space around $v \in \mathrm{X}_v$, then the ideal of $\mathrm{X}_\mathrm{I}|_{\mathrm{U}_v}$ is a direct sum of the T-eigenspaces consisting of those functions on U_v with torus weight contained in

$$\mathrm{S} := \Big(\mathrm{span}_{\mathbf{Z}_{\leqslant 0}}(\Delta^+ \setminus \Delta^+_\mathrm{I})\Big) \setminus \{0\}.$$

(*c*) Let φ_v be the induced inclusion $\varphi_v \colon \mathrm{X}_\mathrm{I} \longrightarrow \mathrm{X}_v$ (i.e., ψ_v considered as a map to X_v). For any dominant weight λ, the pullback map $\mathrm{H}^0(\mathrm{X}_\mathrm{I}, \varphi_v^*(\mathrm{L}_\lambda|_{\mathrm{X}_v})) \xleftarrow{\varphi_v^*} \mathrm{H}^0(\mathrm{X}_v, \mathrm{L}_\lambda|_{\mathrm{X}_v})$ is surjective, and $\mathrm{H}^0(\mathrm{X}_\mathrm{I}, \varphi_v^*(\mathrm{L}_\lambda|_{\mathrm{X}_v})) = \mathrm{V}_{\overline{\mu}} \otimes \mathbf{C}_{\mu'}$ as a representation of $\mathfrak{g}_\mathrm{I}$, where $\overline{\mu}$ and μ' are the restrictions of $-v\lambda$ to $\bar{\mathfrak{t}}$ and $\mathfrak{a}$ respectively under the decomposition $\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathfrak{a}$ from §2.3.

Proof. Two elements g_1, g_2 of G_I have the same image under the composite map if and only if there is a $b \in \mathrm{B}$ such that $g_1\dot{v} = g_2\dot{v}b$, i.e., $g_2^{-1}g_1 = \dot{v}b\dot{v}^{-1}$, or equivalently, if g_1 and g_2 are in the same coset of the subgroup $\mathrm{H} := \mathrm{G}_\mathrm{I} \cap \dot{v}\mathrm{B}\dot{v}^{-1}$. Let $\mathrm{H}_\circ$ be the connected component of the identity of H. Since G_I and $\dot{v}\mathrm{B}\dot{v}^{-1}$ both contain T, $\mathrm{H}_\circ$ is determined by its torus weights on the tangent space at the identity. For every root $\alpha \in \Delta$ exactly one of $\pm\alpha$ is a root of $\dot{v}\mathrm{B}\dot{v}^{-1}$, and so $\mathrm{H}_\circ$ must be a Borel subgroup of G_I. This implies that $\mathrm{H} = \mathrm{H}_\circ$, since $\mathrm{H}_\circ$ is normal in H and since every Borel subgroup of G_I is its own normalizer. The roots of $\dot{v}\mathrm{B}\dot{v}^{-1}$ are $v\Delta^+ = -\Phi_{v^{-1}} \sqcup \Phi^c_{v^{-1}}$; by hypothesis $\Delta^+_\mathrm{I} \subseteq \Phi_{v^{-1}}$ and so $\mathrm{H}_\circ$ must contain $\mathrm{B}_\mathrm{I}^{\mathrm{op}}$. Thus $\mathrm{H}_\circ = \mathrm{B}_\mathrm{I}^{\mathrm{op}}$ and the image of G_I under the composite map is X_I. The induced map ψ_v is independent of the lift of v since $\mathrm{T} \subseteq \mathrm{G}_\mathrm{I}$, and it is clear from the description that ψ_v is G_I-equivariant. This proves (*a*).

Let U_v be the affine space $\mathrm{B}\dot{v}\mathrm{B}/\mathrm{B}$. Under the composite map from G_I to X inducing ψ_v, the image $\mathrm{U}_{\mathrm{I},v} := \mathrm{B}_\mathrm{I}\dot{v}\mathrm{B}/\mathrm{B}$ of B_I forms an open cell of $\psi_v(\mathrm{X}_\mathrm{I})$ around $v \in \psi_v(\mathrm{X}_\mathrm{I})$. Since $\mathrm{B}_\mathrm{I} \subseteq \mathrm{B}$ this shows that $\mathrm{U}_{\mathrm{I},v}$ is contained in U_v and hence, taking Zariski closures in X, that $\psi_v(\mathrm{X}_\mathrm{I})$ is contained in X_v.

By the above discussion on open affine cells, $\mathrm{U}_v = \mathrm{Spec}(\mathbf{C}[z_{-\alpha}]_{\alpha\in\Phi_{v^{-1}}})$ where each $z_{-\alpha}$ is an independent variable on which T acts via the weight $-\alpha$. Similarly $\mathrm{U}_{\mathrm{I},v} = \mathrm{Spec}(\mathbf{C}[z'_{-\alpha}]_{\alpha\in\Delta^+_\mathrm{I}})$ where again each $z'_{-\alpha}$ is an independent variable on which T acts via the weight $-\alpha$. The T-equivariant closed embedding $\mathrm{U}_{\mathrm{I},v} \hookrightarrow \mathrm{U}_v$ corresponds to a T-equivariant surjective map of rings $\mathbf{C}[z_{-\alpha}]_{\alpha\in\Phi_{v^{-1}}} \longrightarrow \mathbf{C}[z'_{-\alpha}]_{\alpha\in\Delta^+_\mathrm{I}}$. If γ is a weight in $\mathrm{span}_{\mathbf{Z}_{\leqslant 0}}\,\mathrm{I} = \mathrm{span}_{\mathbf{Z}_{\leqslant 0}}\,\Delta^+_\mathrm{I}$ then the dimension of the T-eigenspace of weight γ in both rings is the same. In particular, no monomial in the variables $\{z_{-\alpha}\}_{\alpha\in\Delta^+_\mathrm{I}}$ is in the kernel of the map, while all monomials involving the variables $\{z_{-\alpha}\}_{\alpha\in\Phi_{v^{-1}}\setminus\Delta^+_\mathrm{I}}$ are. Therefore the kernel of the surjection is the direct sum of the T-eigenspaces consisting of the functions whose weight lies in S. This proves (*b*).

If λ is dominant then L_λ is basepoint free on X, and so the pullback map ψ_v^* from $\mathrm{H}^0(\mathrm{X}, \mathrm{L}_\lambda)$ to $\mathrm{H}^0(\mathrm{X}_\mathrm{I}, \psi_v^*\mathrm{L}_\lambda) = \mathrm{H}^0(\mathrm{X}_\mathrm{I}, \varphi_v^*(\mathrm{L}_\lambda|_{\mathrm{X}_v}))$ is nonzero. On the other hand, by part (*a*) the pullback map ψ_v^* is G_I-equivariant, and since $\mathrm{H}^0(\mathrm{X}_\mathrm{I}, \psi_v^*\mathrm{L}_\lambda)$ is an irreducible representation of G_I, ψ_v^* must be surjective. The map φ_v^* is therefore also surjective since ψ_v^* factors through φ_v^*. Under the composite map $\mathrm{G}_\mathrm{I} \longrightarrow \mathrm{X}_\mathrm{I} \xrightarrow{\varphi_v} \mathrm{X}$ the point $1_{\mathrm{G}_\mathrm{I}} \in \mathrm{G}_\mathrm{I}$ gets sent to $v \in \mathrm{X}_v$, and hence the torus weight of $\varphi_v^*\mathrm{L}_\lambda$ at the image of $1_{\mathrm{G}_\mathrm{I}}$ in X_I is $-v\lambda$, and therefore $\mathrm{H}^0(\mathrm{X}_\mathrm{I}, \varphi_v^*(\mathrm{L}_\lambda|_{\mathrm{X}_v})) \cong \mathrm{V}_{\overline{\mu}} \otimes \mathbf{C}_{\mu'}$ as representations of $\mathfrak{g}_\mathrm{I}$ by Theorem 2.4.4, proving (*c*). □

We will also need a variant of Proposition 2.5.2(*a,c*) under the "opposite" hypothesis that $\Delta_{\mathrm{I}}^{+} \cap \Phi_{v^{-1}} = \emptyset$. We omit the demonstration since it only involves minor modifications of the proof of Proposition 2.5.2.

Proposition (2.5.3) — Let v be an element of the Weyl group such that that $\Delta_{\mathrm{I}}^{+} \cap \Phi_{v^{-1}} = \emptyset$. Then the map $\mathrm{G_I} \longrightarrow \mathrm{G}$ defined by $g \mapsto gv$ induces a $\mathrm{G_I}$-equivariant embedding $\psi'_v \colon \mathrm{G_I}/\mathrm{B_I} \longrightarrow \mathrm{X}$ sending $1_{\mathrm{G_I}}$ to $v \in \mathrm{X}$. For any dominant weight λ, $\mathrm{H}^0(\mathrm{G_I}/\mathrm{B_I}, \psi_v'^{*}\mathrm{L}_\lambda) \cong \mathrm{V}_{\overline{\mu}}^{*} \otimes \mathbf{C}_{\mu'}$ as representations of $\mathfrak{g}_{\mathrm{I}}$, where where $\overline{\mu}$ and μ' are the restrictions of $-v\lambda$ to $\bar{\mathfrak{t}}$ and $\mathfrak{a}$ respectively under the decomposition $\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathfrak{a}$ from §2.3.

The action of $\mathrm{G_I}$ on $\mathrm{X_I}$ factors through the center and so $\overline{\mathrm{G}}$ acts naturally on $\mathrm{X_I}$. As a $\overline{\mathrm{G}}$-variety $\mathrm{X_I}$ (and $\mathrm{G_I}/\mathrm{B_I}$) are isomorphic in a unique way to $\overline{\mathrm{X}} := \overline{\mathrm{G}}/\overline{\mathrm{B}}$, and in the statement of the main theorem we will also use ψ_v and ψ'_v for the maps from $\overline{\mathrm{X}}$ into X given by the constructions in Propositions 2.5.2 and 2.5.3.

Construction of $\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}$ **and maps.** For any sequence $\underline{v} = (v_1, \ldots, v_{k+1})$ of Weyl group elements we set $\mathrm{X}_{\underline{v}} := \mathrm{X}_{v_1} \times \cdots \times \mathrm{X}_{v_{k+1}}$ and consider it as a B-variety where B acts diagonally. We define $\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}$ to be the quotient $\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}} := (\mathrm{G} \times \mathrm{X}_{\underline{v}})/\mathrm{B}$ where the B-action is given by

$$b \cdot (g, x_1, \ldots, x_{k+1}) = (gb^{-1}, b \cdot x_1, \ldots, b \cdot x_{k+1})$$

for a point $(g, x_1, \ldots, x_{k+1})$ of $\mathrm{G} \times \mathrm{X}_{v_1} \times \cdots \times \mathrm{X}_{v_{k+1}}$.

The group G acts on $\mathrm{G} \times \mathrm{X}_{\underline{v}}$ by left multiplication on the first factor. Since this action commutes with the action of B above it descends to an action of G on $\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}$. The map from $\mathrm{G} \times \mathrm{X}_{\underline{v}}$ to X^{k+1} given by

(2.5.4) $$(g, x_1, \ldots, x_{k+1}) \mapsto (g \cdot x_1, \ldots, g \cdot x_{k+1})$$

is invariant under the B-action. If we let G act on X^{k+1} diagonally then (2.5.4) is also G-equivariant and hence descends to a G-equivariant morphism $f_{\underline{v}} \colon (\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}) \longrightarrow \mathrm{X}^{k+1}$.

Similarly, the map $\mathrm{G} \times \mathrm{X}_{\underline{v}} \longrightarrow \mathrm{G}$ given by projection onto the first factor descends to a G-equivariant map $f_\circ \colon (\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}) \longrightarrow \mathrm{X}$ expressing $\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}$ as an $\mathrm{X}_{\underline{v}}$-bundle over X. In particular, setting $\mathrm{N} = \dim(\mathrm{X}) = |\Delta^+|$, we obtain that $\dim(\mathrm{X}_{\underline{v}}) = \mathrm{N} + \sum_{i=1}^{k+1} \ell(v_i)$, and hence $\dim(\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}) = \dim(\mathrm{X}^{k+1})$ if and only if $\sum_{i=1}^{k+1} \ell(v_i) = k\mathrm{N}$.

Proposition (2.5.5) — If $\underline{v} = (v_1, \ldots, v_{k+1})$ and $\sum_{i=1}^{k+1} \ell(v_i) = k\mathrm{N}$ then the degree of $f_{\underline{v}} \colon (\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}) \longrightarrow \mathrm{X}^{k+1}$ is given by the intersection number $\cap_{i=1}^{k+1} [\Omega_{w_0 v_i^{-1}}] = \cap_{i=1}^{k+1} [\mathrm{X}_{v_i^{-1}}]$.

Proof. After re-indexing k as $k+1$, this is [DR1, Corollary (3.7.5)], along with the observation that the variety $\mathrm{Q}_{\underline{\mathbf{v}}}$ used in the corollary is our variety $\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}$, and the map $h \colon \mathrm{Q}_{\underline{\mathbf{v}}} \longrightarrow \mathrm{X}^{k+1}$ considered there is our map $f_{\underline{v}}$. □

2.6. The Littlewood-Richardson Cone. For any $k \geqslant 1$, let $\mathcal{C}(k)$ be the *Littlewood-Richardson cone*, i.e., the rational cone generated by the tuples $(\mu_1, \ldots, \mu_k, \mu)$ of dominant weights such that V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. It is known that $\mathcal{C}(k)$ is polyhedral. A face of $\mathcal{C}(k)$ is called *regular* if it intersects the locus of strictly dominant weights.

Description of regular faces. For any set I of simple roots, we define P_{I} to be the parabolic subgroup associated to I. For any parabolic $\mathrm{P} \supseteq \mathrm{B}$ we denote the Weyl group of P by $\mathcal{W}_{\mathrm{P}}$.

For a set I of simple roots we wish to consider elements $w_1,\ldots, w_k$, and w of $\mathcal{W}$ satisfying the following conditions (first identified by Belkale-Kumar in [BK]) with respect to I:

(2.6.1)
- (*i*) Each w_i is of minimal length in the coset $w_i\mathcal{W}_{\mathrm{P}_{\mathrm{I}}}$, and w is of minimal length in the coset $w\mathcal{W}_{\mathrm{P}_{\mathrm{I}}}$.
- (*ii*) $\ell(w) = \sum_{i=1}^{k} \ell(w_i)$ and $\cap_{i=1}^{k}[\Omega_{w_i}] \cdot [\mathrm{X}_w] = 1$.
- (*iii*) The weight $\sum_{i=1}^{k} w_i^{-1} \cdot 0 - w^{-1} \cdot 0$ belongs to $\mathrm{span}_{\mathbf{Z}_{\geqslant 0}}$ I,

where $u \cdot 0$ denotes the affine action of an element $u \in \mathcal{W}$ on the zero weight. To produce examples of such $w_1,\ldots, w_k$, and w it is usually easier to use the following equivalent formulation of conditions (2.6.1):

(2.6.2)
- (*i*) The classes $[\Omega_{w_i}]$, $i = 1,\ldots, k$, and $[\Omega_w]$ are pullbacks of Schubert classes σ_i, $i = 1,\ldots, k$ and σ respectively from $\mathrm{G}/\mathrm{P}_{\mathrm{I}}$.
- (*ii*) The coefficient of σ when writing the product $\cap_{i=1}^{k}\sigma_i$ as a sum of basis elements is 1.
- (*iii*) The weight $\sum_{i=1}^{k} w_i^{-1} \cdot 0 - w^{-1} \cdot 0$ belongs to $\mathrm{span}_{\mathbf{Z}_{\geqslant 0}}$ I.

The conditions above are directly equivalent, i.e., (2.6.2)(*i*) is equivalent to (2.6.1)(*i*) and (2.6.2)(*ii*) is equivalent to (2.6.1)(*ii*).

The work of Ressayre gives an explicit description of the regular faces of $\mathcal{C}(k)$. The following is a translation of [R, Theorem D] into our notation:

Theorem (2.6.3) —

(*a*) Let I be a set of simple roots and $w_1,\ldots, w_k$, and w elements of $\mathcal{W}$ satisfying conditions (2.6.1) with respect to I. Then the set

$$\left\{(\mu_1, \ldots, \mu_k, \mu) \in \mathcal{C}(k) \mid \sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu \in \mathrm{span}_{\mathbf{Q}_{\geqslant 0}} \mathrm{I}\right\}$$

is a regular face of codimension $(n - |\mathrm{I}|)$ of $\mathcal{C}(k)$. Here $|\mathrm{I}|$ denotes the cardinality of the set I and n the rank of G.

(*b*) Any regular face of $\mathcal{C}(k)$ is of the form given in part (*a*).

The theorem of Ressayre above is not necessary for the proof of the reduction theorem. Its importance for this paper is that it links the combinatorial conditions used in the proof with the geometry of the Littlewood-Richardson cone, and guarantees that there are examples to which the reduction rules apply.

Reduction Theorem (3.1.1) — Suppose that we are given a set I of simple roots and elements $w_1, \ldots, w_k, w \in \mathcal{W}$ satisfying conditions (2.6.1)(*i*,*ii*) with respect to I. Let $\overline{\mathrm{G}}$ be the semisimple part of $\mathrm{P_I}$, $\overline{\mathrm{X}} = \overline{\mathrm{G}}/\overline{\mathrm{B}}$, $\mathrm{X} = \mathrm{G}/\mathrm{B}$, and $\psi := \psi_{w_1^{-1}w_0} \times \cdots \times \psi_{w_k^{-1}w_0} \times \psi'_{w^{-1}}$ the $\overline{\mathrm{G}}$-equivariant map $\overline{\mathrm{X}}^{k+1} \longrightarrow \mathrm{X}^{k+1}$ given by the constructions in Propositions 2.5.2 and 2.5.3. Suppose that dominant weights $\mu_1, \ldots, \mu_k$, and μ satisfy

$$\textbf{(3.1.2)} \qquad \sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu \in \mathrm{span}_{\mathbf{Q}}\, \mathrm{I},$$

and let $\overline{\mu}_1, \ldots, \overline{\mu}_k$, and $\overline{\mu}$ be the reductions (cf. §2.3) of $w_1^{-1}\mu_1, \ldots, w_k^{-1}\mu_k$, and $w^{-1}\mu$ respectively to $\overline{\mathrm{T}}$. Set $\mathrm{L} := \mathrm{L}_{-w_0\mu_1} \boxtimes \cdots \boxtimes \mathrm{L}_{-w_0\mu_k} \boxtimes \mathrm{L}_{\mu}$ on X^{k+1}. Then the pullback of global sections of L by ψ induces an isomorphism of vector spaces

$$(\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_{\mu}^*)^{\mathrm{G}} \xrightarrow{\sim} (\mathrm{V}_{\overline{\mu}_1} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*)^{\overline{\mathrm{G}}},$$

and in particular $\mathrm{mult_G}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}) = \mathrm{mult}_{\overline{\mathrm{G}}}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k})$.

Proof. We will construct a sequence of isomorphisms of vector spaces starting with $(\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_{\mu}^*)^{\mathrm{G}}$ and ending with $(\mathrm{V}_{\overline{\mu}_1} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*)^{\overline{\mathrm{G}}}$. Afterwards we will check that the composite isomorphism is that induced by pullback of global sections via ψ.

Step 1. Set $\lambda_i = -w_0\mu_i$ for $i = 1, \ldots, k$, and $\lambda_{k+1} = \mu$. Let L be the line bundle $\mathrm{L}_{\lambda_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\lambda_{k+1}}$ on X^{k+1} as above so that $\mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L}) = \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_{\mu}^*$.

Set $v_i = w_i^{-1}w_0$ for $i = 1, \ldots, k$, $v_{k+1} = w^{-1}$, and $\underline{v} = (v_1, \ldots, v_{k+1})$ and consider the map $f_{\underline{v}} \colon (\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}) \longrightarrow \mathrm{X}^{k+1}$ from §2.5. By Proposition 2.5.5 the degree of $f_{\underline{v}}$ is given by

$$\bigcap_{i=1}^{k+1} [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [\mathrm{X}_w] \overset{(2.6.1)(ii)}{=} 1,$$

and therefore $f_{\underline{v}}$ is a proper birational map. Since X^{k+1} is smooth it follows that $f_{\underline{v}*}(f_{\underline{v}}^*\mathrm{L}) = \mathrm{L}$ and therefore pullback induces an isomorphism

$$\mathrm{H}^0(\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L}) \xleftarrow{f_{\underline{v}}^*} \mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L}).$$

Because $f_{\underline{v}}$ is G-equivariant, $f_{\underline{v}}^*$ induces an isomorphism of G-invariant subspaces, and we may therefore focus our attention on $\mathrm{H}^0(\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L})^{\mathrm{G}}$.

Step 2. Let $\mathcal{E}_2 = f_{\circ *}(f_{\underline{v}}^*\mathrm{L})$, where $f_\circ \colon (\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}) \longrightarrow \mathrm{X}$ is the map from §2.5, and let E_2 be the fibre of $\mathcal{E}_2$ over $e \in \mathrm{X}$. Since $f_\circ$ is also G-equivariant, pushforward induces an isomorphism $\mathrm{H}^0(\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L})^{\mathrm{G}} \xrightarrow{\sim} \mathrm{H}^0(\mathrm{X}, \mathcal{E}_2)^{\mathrm{G}}$. By Principle 2.4.1 restriction to the fibre over e induces an isomorphism $\mathrm{H}^0(\mathrm{X}, \mathcal{E}_2)^{\mathrm{G}} \xrightarrow{\sim} \mathrm{E}_2^{\mathrm{B}}$.

Step 3. The fibre of $f_\circ$ over $e \in \mathrm{X}$ is $\mathrm{X}_{\underline{v}}$. Let $i_{\underline{v}} \colon \mathrm{X}_{\underline{v}} \longrightarrow \mathrm{X}^{k+1}$ be the restriction of $f_{\underline{v}}$ to this fibre. From the construction in §2.5 it follows that $i_{\underline{v}}$ is the product of the natural inclusion maps $\mathrm{X}_{v_j} \hookrightarrow \mathrm{X}$ for $j = 1,\ldots, k+1$. Hence by the theorem on cohomology and base change $\mathrm{E}_2 = \mathrm{H}^0(\mathrm{X}_{\underline{v}}, i_{\underline{v}}^*\mathrm{L})$. Set $\gamma = \sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu$; then γ is the weight of T acting on $i_{\underline{v}}^*\mathrm{L}$ at the point $p := (v_1, \ldots, v_{k+1}) \in \mathrm{X}_{\underline{v}}$, and $\gamma \in \operatorname{span}_{\mathbf{Q}} \mathrm{I}$ by condition (3.1.2).

Let $\mathrm{U} = \mathrm{U}_{v_1} \times \cdots \times \mathrm{U}_{v_{k+1}}$ be the product of the B-stable affine spaces U_{v_i} from §2.5; the origin of U is the point p. Since U is open in the irreducible variety $\mathrm{X}_{\underline{v}}$, restriction gives an B-equivariant inclusion $\mathrm{E}_2 = \mathrm{H}^0(\mathrm{X}_{\underline{v}}, i_{\underline{v}}^*\mathrm{L}|_{\mathrm{X}_{\underline{v}}}) \hookrightarrow \mathrm{H}^0(\mathrm{U}, i_{\underline{v}}^*\mathrm{L}|_{\mathrm{U}})$.

Set $\mathrm{L}_\mathrm{U} = (i_{\underline{v}}^*\mathrm{L})|_\mathrm{U}$. Since U is isomorphic to affine space, L_U is (non-equivariantly) trivial on U. Let $s_\circ$ be a section of L_U which is nowhere vanishing. The torus T takes $s_\circ$ to another nowhere vanishing section which must therefore be a multiple of $s_\circ$, i.e., T acts on $s_\circ$ via a weight. This must be the same as the weight of the action of L_U at p, and so T acts on $s_\circ$ with weight γ. Let B^+ be the unipotent radical of B. By the same reasoning, B^+ must take $s_\circ$ to a multiple of itself. Since B^+ has only the trivial one-dimensional representation $s_\circ$ must be fixed by B^+.

Every section $s \in \mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})$ can be written as $s = s_\circ h$ for some function $h \in \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})$. The section s is B-invariant if and only if h is B^+-invariant and h is an eigenfunction of T on which T acts via $-\gamma$. For any weight δ, let $\mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_\delta$ denote the space of eigenfunctions of T on which T acts via δ. Let $\mathfrak{b}^+$ be the Lie algebra of B^+ (i.e, the nilpotent radical of $\mathfrak{b}$); $\mathfrak{b}^+$ acts on $\mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})$ via derivations. By the correspondence above between sections of L_U and functions on U we have $\mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})^\mathrm{B} = \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma}^{\mathfrak{b}^+}$.

For each $\beta \in \Delta^+$ let ∂_β be a vector field giving the action of a nonzero element of $\mathfrak{g}^\beta \subseteq \mathfrak{b}^+$ on U. Each ∂_β is a graded first-order differential operator of degree β, i.e.,

$$\partial_\beta \left(\mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_\delta\right) \subseteq \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{\delta+\beta}$$

for each weight δ. Thus, we obtain

$$\textbf{(3.1.3)}\quad \mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})^\mathrm{B} = \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma}^{\mathfrak{b}^+} = \bigcap_{\beta\in\Delta^+} \ker\left(\mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma} \xrightarrow{\partial_\beta} \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma+\beta}\right).$$

By repeating the same argument with the subgroup B_I we obtain a similar identification

$$\textbf{(3.1.4)}\quad \mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})^{\mathrm{B}_\mathrm{I}} = \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma}^{\mathfrak{b}_\mathrm{I}^+} = \bigcap_{\beta\in\Delta_\mathrm{I}^+} \ker\left(\mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma} \xrightarrow{\partial_\beta} \mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma+\beta}\right).$$

Since $\gamma \in \operatorname{span}_{\mathbf{Q}} \mathrm{I}$, if $\beta \in \Delta^+ \setminus \Delta_\mathrm{I}^+$ then $-\gamma + \beta \notin \operatorname{span}_{\mathbf{Z}_{\leqslant 0}} \Delta^+$ and so $\mathrm{H}^0(\mathrm{U}, \mathcal{O}_\mathrm{U})_{-\gamma+\beta} = 0$ by Lemma 2.5.1. Thus the right-hand sides of (3.1.3) and (3.1.4) are equal, and hence $\mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})^\mathrm{B} = \mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})^{\mathrm{B}_\mathrm{I}}$. Since the inclusion map $\mathrm{E}_2 \hookrightarrow \mathrm{H}^0(\mathrm{U}, \mathrm{L}_\mathrm{U})$ is B-equivariant we conclude that $\mathrm{E}_2^\mathrm{B} = \mathrm{E}_2^{\mathrm{B}_\mathrm{I}}$. Passing to the Lie algebra of B_I we are reduced to studying $\mathrm{E}_2^{\mathfrak{b}_\mathrm{I}}$.

Step 4. An element $u \in \mathcal{W}$ is of minimal length in the coset $u\mathcal{W}_{\mathrm{P}_\mathrm{I}}$ if and only if $\Delta_\mathrm{I}^+ \cap \Phi_u = \emptyset$. Applying this observation to each w_i, we conclude that $\Delta_\mathrm{I}^+ \subseteq \Phi_{w_i}^\mathrm{c} = \Phi_{v_i^{-1}}$, and hence by Proposition 2.5.2(*b*) we have B_I-equivariant embeddings $\varphi_{v_i} \colon \mathrm{X}_\mathrm{I} \longrightarrow \mathrm{X}_{v_i}$ for $i = 1,\ldots, k$.

The variety $X_{v_{k+1}}$ is stable under B and hence under the subgroup $B_I \subseteq B$. The stabilizer subgroup of $v_{k+1} \in X$ is $v_{k+1}Bv_{k+1}^{-1}$ with roots $v_{k+1}\Delta^+ = w^{-1}\Delta^+ = -\Phi_w \sqcup \Phi_w^c$. Applying the observation on minimality of length to w we conclude that $\Delta_I^+ \subseteq \Phi_w^c$, and hence that $B_I \subseteq v_{k+1}Bv_{k+1}^{-1}$, i.e., B_I fixes the point $v_{k+1} \in X_{v_{k+1}}$. Let $j_{v_{k+1}}\colon \operatorname{Spec}(\mathbf{C}) \longrightarrow X_{v_{k+1}}$ be the B_I-equivariant inclusion of the point v_{k+1}.

Finally, let $\varphi_{\underline{v}}\colon X_I^k \longrightarrow X_{\underline{v}}$ be the map including $X_I^k = X_I^k \times \operatorname{Spec}(\mathbf{C})$ into $X_{\underline{v}}$ via the product inclusions $\varphi_{v_1} \times \cdots \times \varphi_{v_k} \times j_{k+1}$ and set $E_3 = H^0(X_I^k, \varphi_{\underline{v}}^* i_{\underline{v}}^* L)$. By the Kunnuth theorem

$$\textbf{(3.1.5)} \qquad E_3 = \left(\bigotimes_{i=1}^{k} H^0(X_I, \varphi_{v_i}^*(L_{\lambda_i}|_{X_{v_i}}))\right) \otimes \left(j_{k+1}^*(L_{\lambda_{k+1}}|_{X_{v_{k+1}}})\right).$$

By Proposition 2.5.2(*c*) each of the pullback maps

$$\varphi_{v_i}^*\colon H^0(X_{v_i}, L_{\lambda_i}|_{X_{v_i}}) \longrightarrow H^0(X_I, \varphi_{v_i}^*(L_{\lambda_i}|_{X_{v_i}}))$$

are surjective for $i = 1,\ldots, k$, and certainly $j_{k+1}^*\colon H^0(X_{v_{k+1}}, L_{\lambda_{k+1}}|_{X_{v_{k+1}}}) \longrightarrow j_{k+1}^*(L_{\lambda_{k+1}}|_{v_{k+1}})$ is surjective since $L_{\lambda_{k+1}}$ is basepoint free on X and the pullback is to a point. Thus the B_I-equivariant pullback map $\varphi_{\underline{v}}^*\colon E_2 \longrightarrow E_3$ is surjective. We want to see that this surjection induces an isomorphism of $\mathfrak{b}_I$-invariants.

Let E_1 be the kernel of the surjection above. If $\mathcal{I}$ is the ideal sheaf of $\varphi_{\underline{v}}(X_I^k)$ in $X_{\underline{v}}$ then $E_1 = H^0(X_{\underline{v}}, (i_{\underline{v}}^*L) \otimes_{\mathcal{O}_{X_{\underline{v}}}} \mathcal{I})$. As in step 3 we will analyze E_1 via the inclusion $E_1 \hookrightarrow H^0(U, L_U \otimes_{\mathcal{O}_U} \mathcal{I}|_U)$ obtained by restriction to U. As in step 3 every section $s \in H^0(U, L_U \otimes_{\mathcal{O}_U} \mathcal{I}|_U)$ can be written as $s_\circ h$ with $h \in H^0(U, \mathcal{I}|_U)$. Since X_I^k is a product subvariety in the product variety $X_{\underline{v}}$, and U is a product subset, the ideal $H^0(U, \mathcal{I}|_U)$ is the sum of the pullbacks to U of the ideals of $X_I|_{U_{v_i}}$, $i = 1,\ldots, k$ and the ideal of the point $v_{k+1} \in U_{v_{k+1}}$. By Proposition 2.5.2(*b*) for each $i = 1,\ldots, k$, the ideal of $X_I|_{U_{v_i}}$ consists of the direct sum of the T-eigenspaces of functions on U_{v_i} with torus weights contained in $S = \left(\operatorname{span}_{\mathbf{Z}_{\leqslant 0}}(\Delta^+ \setminus \Delta_I^+)\right) \setminus \{0\}$. Now $U_{v_{k+1}} = \operatorname{Spec}(\mathbf{C}[z_{-\alpha}]_{\alpha\in\Phi_{v_{k+1}^{-1}}})$ and the ideal of v_{k+1} in $U_{v_{k+1}}$ is generated by $\{z_{-\alpha}\}_{\alpha\in\Phi_{v_{k+1}^{-1}}}$. Since $\Phi_{v_{k+1}^{-1}} = \Phi_w$, and again using the observation on the minimality of w, we conclude that the weights of all T-eigenfunctions in the ideal of v_{k+1} in $U_{v_{k+1}}$ are also contained in S. Pulling back these ideals to U, and using the fact that T acts on $s_\circ$ with weight $\gamma \in \operatorname{span}_{\mathbf{Q}} I$, we conclude that all T-eigensections $s = s_\circ h \in H^0(U, L_U \otimes_{\mathcal{O}_U} \mathcal{I}|_U)$ have weights outside $\operatorname{span}_{\mathbf{Z}} I$. Since $E_1 \hookrightarrow H^0(U, L_U \otimes_{\mathcal{O}_U} \mathcal{I}|_U)$ is a B_I-equivariant inclusion, we conclude that the same is true for the weights of E_1. In particular, no weight of E_1 is contained in $\{0\} \cup \Delta_I^+$. Thus by Lemma 2.3.2 the surjection $E_2 \longrightarrow E_3$ induces an isomorphism $E_2^{\mathfrak{b}_I} \xrightarrow{\sim} E_3^{\mathfrak{b}_I}$.

Step 5. By Lemma 2.5.2(*c*), for $i = 1,\ldots, k$ we have

$$H^0(X_I, \varphi_{v_i}^*(L_{\lambda_i}|_{X_{v_i}})) \cong V_{\overline{\mu}_i} \otimes \mathbf{C}_{\mu_i'}$$

as $\mathfrak{g}_I$-modules and the $\mathfrak{b}_I$-module structure on $H^0(X_I, \varphi_{v_i}^*(L_{\lambda_i}|_{X_{v_i}}))$ is simply the restriction of the $\mathfrak{g}_I$-module structure. Here $\overline{\mu}_i$ and μ_i' are restrictions to $\bar{\mathfrak{t}}$ and $\mathfrak{a}$ respectively of $w_i^{-1}\mu_i = -v_i\lambda_i$ using the decomposition $\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathfrak{a}$ from §2.3.

Similarly, the one-dimensional $\mathfrak{t}$-representation $j_{k+1}^*(L_{\lambda_{k+1}}|_{X_{v_{k+1}}})$ decomposes as $\mathbf{C}_{-\overline{\mu}} \otimes \mathbf{C}_{-\mu'}$ where $\overline{\mu}$ and μ' are the restrictions to $\bar{\mathfrak{t}}$ and $\mathfrak{a}$ respectively of $v_{k+1}\lambda_{k+1} = w^{-1}\mu$.

Thus, using (3.1.5) and collecting the one-dimensional representations of $\mathfrak{a}$ we have

$$\mathrm{E}_3 = \left(\bigotimes_{i=1}^{k} \mathrm{V}_{\overline{\mu}_i}\right) \otimes \mathbf{C}_{-\overline{\mu}} \otimes \mathbf{C}_{\left(\sum_{i=1}^{k} \mu_i'\right)-\mu'}.$$

However, since restriction is a homomorphism, $\left(\sum_{i=1}^{k} \mu_i'\right) - \mu'$ is just the restriction to $\mathfrak{a}$ of the weight γ, and this is zero by Condition (3.1.2) and Lemma 2.3.1. Thus

$$\mathrm{E}_3 = \left(\bigotimes_{i=1}^{k} \mathrm{V}_{\overline{\mu}_i}\right) \otimes \mathbf{C}_{-\overline{\mu}} \tag{3.1.6}$$

and hence E_3 is a $\mathfrak{b}_{\mathrm{I}}$-module with trivial $\mathfrak{a}$-action, i.e., E_3 is really a $\mathfrak{b}_{\mathrm{I}}/\mathfrak{a} = \overline{\mathfrak{b}}$-module and so $\mathrm{E}_3^{\mathfrak{b}_{\mathrm{I}}} = \mathrm{E}_3^{\overline{\mathfrak{b}}}$.

Step 6. It is straightforward to see that $\mathrm{E}_3^{\overline{\mathfrak{b}}} = (\mathrm{V}_{\overline{\mu}_i} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*)^{\overline{\mathrm{G}}}$ which will finish the construction of the isomorphism.

The most direct argument is to notice that the $\overline{\mathfrak{b}}^+$-invariants of E_3 are, by (3.1.6), the highest-weight subspaces of the irreducible components of $\otimes_{i=1}^{k}\mathrm{V}_{\overline{\mu}_i}$ tensored with $\mathbf{C}_{-\overline{\mu}}$, and hence the $\overline{\mathfrak{b}}$-invariants of E_3 are the subspace of highest-weight vectors of weight $\overline{\mu}$ in $\otimes_{i=1}^{k}\mathrm{V}_{\overline{\mu}_i}$, which is precisely the subspace $(\mathrm{V}_{\overline{\mu}_i} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*)^{\overline{\mathrm{G}}}$.

A more geometric approach, inducing the isomorphism of vector spaces directly, is to let $\mathcal{E}_3$ be the vector bundle on $\overline{\mathrm{X}} = \overline{\mathrm{G}}/\overline{\mathrm{B}}$ whose fibre over $e \in \overline{\mathrm{X}}$ is E_3. By Principle 2.4.1 $\mathrm{E}_3^{\overline{\mathfrak{b}}} = \mathrm{E}_3^{\overline{\mathrm{B}}} = \mathrm{H}^0(\overline{\mathrm{X}}, \mathcal{E}_3)^{\overline{\mathrm{G}}}$. Equation (3.1.6) shows that $\mathcal{E}_3 = (\otimes_{i=1}^{k} \mathrm{V}_{\overline{\mu}_i}) \otimes_{\mathcal{O}_{\overline{\mathrm{X}}}} \mathrm{L}_{\overline{\mu}}$, and hence

$$\mathrm{H}^0(\overline{\mathrm{X}}, \mathcal{E}_3) = \left(\bigotimes_{i=1}^{k} \mathrm{V}_{\overline{\mu}_i}\right) \otimes \mathrm{H}^0(\overline{\mathrm{X}}, \mathrm{L}_{\overline{\mu}}) = \mathrm{V}_{\overline{\mu}_i} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*$$

by the Borel-Weil theorem. Taking $\overline{\mathrm{G}}$-invariants finishes the alternate argument for Step 6 and the construction of the isomorphism.

Composition of steps 1–6. Finally, we want to check that the composite isomorphism is that induced by pullback via ψ. Recall that we are identifying $\overline{\mathrm{X}}$ and X_{I} by the unique isomorphism respecting their structure as $\overline{\mathrm{G}}$-varieties. Let $\overline{\mathrm{L}} = \psi^*\mathrm{L}$. It is straightforward to check (c.f. Propositions 2.5.2 and 2.5.3) that $\mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \overline{\mathrm{L}}) = \mathrm{V}_{\overline{\mu}_1} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*$. Let $\pi\colon \overline{\mathrm{X}}^{k+1} \longrightarrow \overline{\mathrm{X}}$ be projection onto the final factor. Pushing forward by π we obtain $\mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \overline{\mathrm{L}}) = \mathrm{H}^0(\overline{\mathrm{X}}, \pi_*\overline{\mathrm{L}})$. The main point is that $\pi_*\overline{\mathrm{L}} = \mathcal{E}_3$ and that the pullback map ψ^* on global sections induces the isomorphism $\mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}} \xrightarrow{\sim} \mathrm{H}^0(\mathrm{X}, \mathcal{E}_3)^{\overline{\mathrm{G}}}$ obtained by combining steps 1 through 6.

To see this, let $\overline{\mathrm{X}}^k$ be the fibre of π over $\overline{e} \in \overline{\mathrm{X}}$, and let $j\colon \overline{\mathrm{X}}^k \longrightarrow \overline{\mathrm{X}}^{k+1}$ be the inclusion of this fibre in $\overline{\mathrm{X}}^{k+1}$. By the theorem on cohomology and base change, the fibre of $\pi_*\overline{\mathrm{L}}$ over $\overline{e}$ is equal to $\mathrm{H}^0(\overline{\mathrm{X}}^k, \overline{\mathrm{L}}|_{\overline{\mathrm{X}}^k})$. By a straightforward check the composite map $\psi \circ j$ is equal to $i_{\underline{v}} \circ \varphi_{\underline{v}}$ and hence $\mathrm{H}^0(\overline{\mathrm{X}}^k, \overline{\mathrm{L}}|_{\overline{\mathrm{X}}^k}) = \mathrm{H}^0(\overline{\mathrm{X}}^k, \varphi_{\underline{v}}^* i_{\underline{v}}^*\mathrm{L}) = \mathrm{E}_3$ by the definition in step 4. Thus

$\pi_*\overline{\mathrm{L}} = \mathcal{E}_3$. The content of steps 1–5 is that restriction to $\overline{\mathrm{X}}^k$ (i.e., pullback by $\psi \circ j$) induces an isomorphism $\mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}} \cong \mathrm{E}_3^{\overline{\mathrm{B}}}$. Since ψ is $\overline{\mathrm{G}}$-equivariant, G-invariant sections pull back to $\overline{\mathrm{G}}$-invariant sections, and so the composite isomorphism from steps 1–5 factors as

$$\mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}} \xrightarrow{\psi^*} \mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \overline{\mathrm{L}})^{\overline{\mathrm{G}}} \xrightarrow{j^*} \mathrm{H}^0(\overline{\mathrm{X}}^k, \overline{\mathrm{L}}|_{\overline{\mathrm{X}}^k})^{\overline{\mathrm{B}}} = \mathrm{E}_3^{\overline{\mathrm{B}}}.$$

Via the identification $\mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \overline{\mathrm{L}})^{\overline{\mathrm{G}}} = \mathrm{H}^0(\overline{\mathrm{X}}, \mathcal{E}_3)^{\overline{\mathrm{G}}}$ the map induced by j^* is simply the natural restriction $\mathrm{H}^0(\overline{\mathrm{X}}, \mathcal{E}_3)^{\overline{\mathrm{G}}} \longrightarrow \mathrm{E}_3^{\overline{\mathrm{B}}}$, which is an isomorphism by Principle 2.4.1. The isomorphism $\mathrm{E}_3^{\overline{\mathrm{B}}} \cong \mathrm{H}^0(\overline{\mathrm{X}}, \mathcal{E}_3)^{\overline{\mathrm{G}}}$ in step 6 is simply its inverse. Thus the map $\mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}} \longrightarrow \mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \overline{\mathrm{L}})^{\overline{\mathrm{G}}}$ induced by pullback by ψ is the composition of the maps from steps 1–6, and in particular is an isomorphism. This finishes the proof of the reduction theorem. □

Remarks. Note that $w_1,\ldots, w_k$, and w do not have to satisfy (2.6.1)(*iii*) in order to apply the reduction theorem. Without (2.6.1)(*iii*) however it is not clear that there are examples where the reduction rule applies, whereas such examples are guaranteed by Theorem 2.6.3 if all the conditions do hold. In applications of the reduction theorem, it is convenient that one only has to verify the condition $\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu \in \operatorname{span}_{\mathbf{Q}} \mathrm{I}$ and not that the sum is in $\operatorname{span}_{\mathbf{Q}_{\geqslant 0}} \mathrm{I}$.

Corollary (3.1.7) — Suppose that $w_1,\ldots, w_k$, and w satisfy (2.6.1)(*i*) with respect to I, and that $\cap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0$ (i.e., instead of $= 1$). Then for any dominant weights $\mu_1,\ldots, \mu_k$, and μ such that

$$\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu \in \operatorname{span}_{\mathbf{Q}} \mathrm{I},$$

we have $\operatorname{mult}_{\mathrm{G}}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}) \leqslant \operatorname{mult}_{\overline{\mathrm{G}}}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k})$, where $\overline{\mu}_1,\ldots, \overline{\mu}_k$, and $\overline{\mu}$ are the restrictions to $\overline{\mathrm{T}}$ of $w_1^{-1}\mu_1,\ldots, w_k^{-1}\mu_k$ and $w^{-1}\mu$ respectively.

Proof. We repeat the proof of the reduction theorem. The only difference occurs in Step 1, since the map $f_{\underline{v}}$ now may have degree greater than one, and so we can only conclude that $f_{\underline{v}}^*$ induces an inclusion

$$\mathrm{H}^0(\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L})^{\mathrm{G}} \xhookleftarrow{f_{\underline{v}}^*} \mathrm{H}^0(\mathrm{X}^{k+1}, \mathrm{L})^{\mathrm{G}} \xleftarrow{\sim} (\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_\mu^*)^{\mathrm{G}}.$$

Following through the rest of the steps, we obtain an isomorphism

$$\mathrm{H}^0(\mathrm{G} \times^{\mathrm{B}} \mathrm{X}_{\underline{v}}, f_{\underline{v}}^*\mathrm{L})^{\mathrm{G}} \xrightarrow{\sim} (\mathrm{V}_{\overline{\mu}_1} \otimes \cdots \otimes \mathrm{V}_{\overline{\mu}_k} \otimes \mathrm{V}_{\overline{\mu}}^*)^{\overline{\mathrm{G}}}$$

and taking dimensions gives the inequality. □

4. Examples

4.1. In this section we work out a number of explicit examples of reduction rules. The rules in §4.2–4.3 are of type A, and so already covered by the results in [DW] and [KTT] (the rule in §4.3 actually predates those papers – it is due to Griffiths and Harris). However the notation used in those papers is different from ours (the rules are expressed in

GL_n weights, and the combinatorial data describing the regular faces is presented in a different form) and the examples are included partly to compare the two approaches.

To check if a weight is in $\mathrm{span}_{\mathbf{Q}}\,\mathrm{I}$ one simply converts from the basis of fundamental weights to the root basis by multiplying by the inverse transpose of the Cartan matrix, and then checks that the coordinates of all simple roots outside of I are zero. This is mentioned again in the first example, but afterwards we just write out the corresponding condition.

In order to check that (2.6.1)(*iii*) holds, the formula

(4.1.1)
$$w^{-1}\cdot 0 = w^{-1}\rho - \rho = -\sum_{\alpha\in\Phi_w}\alpha$$

is useful. Mostly, however we will also omit the explicit calculation checking this condition. In particular, in type A_n when $|\mathrm{I}| = n-1$, condition (2.6.1)(*iii*) follows from the condition $\sum_i \ell(w_i) = \ell(w)$ in (2.6.1)(*ii*), and so does not need to be checked again.

Because the reduction rules (and the multiplicities) depend only on the type of the group, we will label the examples and mult by the type, the only exception being for examples involving GL_{n+1}. The labelling of the roots follows the usual convention in [B, Chapter VI]. We will use $\alpha_1,\ldots,\alpha_n$ for the simple roots, and $s_1,\ldots,s_n$ for the corresponding simple reflections. After each of the examples we give an explicit instance with strictly dominant weights where the rule applies. By Theorem 2.6.3 such instances always exist.

4.2. An A_5 to $\mathrm{A}_2\times\mathrm{A}_2$ reduction rule. Let G be of type A_5 and $\mathrm{I} = \{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}$ so that $\mathrm{G}/\mathrm{P}_{\mathrm{I}} = \mathrm{Gr}(2,5)$, the Grassmanian of two-planes in $\mathbf{P}^5$. The Schubert basis for $\mathrm{H}^*(\mathrm{Gr}(2,5),\mathbf{Z})$ consists of the classes σ_{a_1,a_2,a_3} with $2\geqslant a_1\geqslant a_2\geqslant a_3\geqslant 0$. In $\mathrm{H}^*(\mathrm{Gr}(2,5),\mathbf{Z})$ we have the well-known cohomology multiplication $\sigma_{1,0,0}\cdot\sigma_{1,0,0} = \sigma_{2,0,0}+\sigma_{1,1,0}$. The pull-back of $\sigma_{1,0,0}$ to $\mathrm{X}=\mathrm{G}/\mathrm{B}$ is $[\Omega_{s_3}]$ and the pullback of $\sigma_{2,0,0}$ to X is $[\Omega_{s_4s_3}]$, so that if we pick $w_1 = w_2 = s_3$ and $w = s_4s_3$ then w_1, w_2, and w satisfy (2.6.1) with respect to I. The group $\overline{\mathrm{G}}$ we are reducing to is of type $\mathrm{A}_2\times\mathrm{A}_2$, obtained by deleting the middle node of the Dynkin diagram for G.

If $\mu_1 = (a_1,a_2,a_3,a_4,a_5)$, $\mu_2 = (b_1,b_2,b_3,b_4,b_5)$, and $\mu = (c_1,c_2,c_3,c_4,c_5)$ then

$$\begin{aligned} w_1^{-1}\mu_1 &= (a_1, a_2+a_3, -a_3, a_3+a_4, a_5)\\ w_2^{-1}\mu_2 &= (b_1, b_2+b_3, -b_3, b_3+b_4, b_5)\\ w^{-1}\mu &= (c_1, c_2+c_3+c_4, -c_3-c_4, c_3, c_4+c_5)\end{aligned}$$

The group $\overline{\mathrm{G}}$ is a product group and we will use "|" to indicate the division of the restricted weight among the two factors. Since we are deleting the middle node of the Dynkin diagram, the restriction is obtained by ignoring the middle coefficients in the formulas above, so that $\overline{\mu}_1 = (a_1, a_2+a_3 \mid a_3+a_4, a_5)$, $\overline{\mu}_2 = (b_1, b_2+b_3 \mid b_3+b_4, b_5)$, and $\overline{\mu} = (c_1, c_2+c_3+c_4 \mid c_3, c_4+c_5)$.

The condition that the point (μ_1, μ_2, μ) lie on the face of $\mathcal{C}(2)$ determined by I and w_1, w_2, and w_3 is that the coefficient of α_3 is zero when writing $w_1^{-1}\mu_1 + w_2^{-1}\mu_2 - w^{-1}\mu$ as a sum of simple roots (with **Q**-coefficients). This is easily computed by multiplying the sum, in the coordinates of the fundamental weights as above, by the inverse transpose of the Cartan matrix for A_5 and looking at the middle coefficient. This coefficient is

$$\frac{1}{2}\Big((a_1 + 2a_2 + a_3 + 2a_4 + a_5) + (b_1 + 2b_2 + b_3 + 2b_4 + b_4) - (c_1 + 2c_2 + c_3 + c_5)\Big),$$

and thus we arrive at our first example of a reduction rule.

Reduction rule: If

$$c_1 + 2c_2 + c_3 + c_5 = (a_1 + 2a_2 + a_3 + 2a_4 + a_5) + (b_1 + 2b_2 + b_3 + 2b_4 + b_4)$$

then $\mathrm{mult}_{\mathrm{A}_5}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{A}_2\times\mathrm{A}_2}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \mathrm{V}_{\overline{\mu}_2})$, where $\overline{\mu}_1$, $\overline{\mu}_2$, and $\overline{\mu}$ are given by the formulas above.

Example: $\mu_1 = (4, 2, 10, 6, 10)$, $\mu_2 = (10, 4, 12, 4, 2)$, $\mu = (10, 22, 1, 1, 25)$, $\overline{\mu}_1 = (4, 12 \mid 16, 10)$, $\overline{\mu}_2 = (10, 16 \mid 16, 2)$, $\overline{\mu} = (10, 24 \mid 1, 26)$; the multiplicity is 10.

In GL_6 weights, the rule has the following form.

Reduction rule: If dominant GL_6 weights $\mu_1 = (a_0, a_1, a_2, a_3, a_4, a_5)$,$\mu_2 = (b_0, b_1, b_2, b_3, b_4, b_5)$ and $\mu = (c_0, c_1, c_2, c_3, c_4, c_5)$ (which we assume satisfy $\sum_i c_i = \sum_i a_i + \sum_i b_i$) also satisfy

$$c_0 + c_1 + c_4 = (a_0 + a_1 + a_3) + (b_0 + b_1 + b_3)$$

then $\mathrm{mult}_{\mathrm{GL}_6}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1}\otimes\mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{GL}_3\times\mathrm{GL}_3}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1}\otimes\mathrm{V}_{\overline{\mu}_2})$ where $\overline{\mu}_1 = (a_0, a_1, a_3 \mid a_2, a_4, a_5)$, $\overline{\mu}_2 = (b_0, b_1, b_3 \mid b_2, b_4, b_5)$, and $\overline{\mu} = (c_0, c_1, c_4 \mid c_2, c_3, c_5)$.

Example: $\mu_1 = (32, 28, 26, 16, 10, 0)$, $\mu_2 = (32, 22, 18, 6, 2, 0)$, $\mu = (60, 51, 28, 26, 25, 2)$, $\overline{\mu}_1 = (32, 28, 16 \mid 26, 10, 0)$, $\overline{\mu}_2 = (32, 22, 6 \mid 18, 2, 0)$, $\overline{\mu} = (60, 51, 25 \mid 28, 26, 2)$; multiplicity is 12.

4.3. An A_n to A_{n-1} reduction rule. Let G be of type A_n and $\mathrm{I} = \{\alpha_2, \alpha_3, \ldots, \alpha_n\}$ so that $\mathrm{G}/\mathrm{P}_\mathrm{I} = \mathbf{P}^n$. We have $\mathrm{H}^*(\mathbf{P}^n, \mathbf{Z}) = \mathbf{Z}[h]/(h^{n+1})$, where $h \in \mathrm{H}^2(\mathbf{P}^n, \mathbf{Z})$ is the hyperplane class. Each h^i ($1 \leqslant i \leqslant n$) pulls back to the class $[\Omega_{s_i s_{i-1}\cdots s_1}]$ in the cohomology ring of X. For any $0 \leqslant i, j, k \leqslant n$ with $i + j = k$ we have the obvious cohomology multiplication $h^i \cdot h^j = h^k$. Setting $w_1 = s_i s_{i-1} \cdots s_1$, $w_2 = s_j s_{j-1} \cdots s_1$, and $w = s_k s_{k-1} \cdots s_1$, then w_1, w_2, and w satisfy (2.6.1) with respect to I. The group $\overline{\mathrm{G}}$ we are reducing to is of type A_{n-1} obtained by deleting the first node in the Dynkin diagram for G.

If $\mu_1 = (a_1, \cdots, a_n)$, $\mu_2 = (b_1, \ldots, b_n)$ and $\mu = (c_1, \cdots, c_n)$ are dominant weights then

$$\begin{aligned} w_1^{-1}\mu_1 &= (-a_1 - a_2 - \cdots - a_i, a_1, a_2, \cdots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \cdots, a_n), \\ w_2^{-1}\mu_2 &= (-b_1 - b_2 - \cdots - b_j, b_1, b_2, \cdots, b_{j-1}, b_j + b_{j+1}, b_{j+2}, \cdots, b_n), \\ w^{-1}\mu &= (-c_1 - c_2 - \cdots - c_k, c_1, c_2, \cdots, c_{k-1}, c_k + c_{k+1}, c_{k+2}, \cdots, c_n). \end{aligned}$$

Restriction to $\overline{\mathrm{G}}$ simply ignores the first entries, so

(4.3.1)
$$\begin{cases} \overline{\mu}_1 &= (a_1, \cdots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \cdots, a_n), \\ \overline{\mu}_2 &= (b_1, \cdots, b_{j-1}, b_j + b_{j+1}, b_{j+2}, \cdots, b_n), \text{ and} \\ \overline{\mu} &= (c_1, \cdots, c_{k-1}, c_k + c_{k+1}, c_{k+2}, \cdots, c_n). \end{cases}$$

Here (and above) coefficients with indices greater than n are assumed to be zero.

Writing $w_1^{-1}\mu_1 + w_2^{-1}\mu_2 - w^{-1}\mu$ as a sum of simple roots and multiplying by $n+1$ to clear denominators, the coefficient of α_1 is

$$(n+1)\sum_{r=i+1}^{n} a_r - \sum_{r=1}^{n} r a_r + (n+1)\sum_{r=j+1}^{n} b_r - \sum_{r=1}^{n} r b_r - (n+1)\sum_{r=k+1}^{n} c_r + \sum_{r=1}^{n} r c_r.$$

Thus we obtain the following family of reduction rules.

Reduction rule: For any integers $0 \leqslant i, j, k \leqslant n$ with $i + j = k$, if dominant weights $\mu_1 = (a_1, \cdots, a_n)$, $\mu_2 = (b_1, \ldots, b_n)$ and $\mu = (c_1, \cdots, c_n)$ satisfy

(4.3.2) $$(n+1)\sum_{r=k+1}^{n} c_r - \sum_{r=1}^{n} r c_r = (n+1)\sum_{r=i+1}^{n} a_r - \sum_{r=1}^{n} r a_r + (n+1)\sum_{r=j+1}^{n} b_r - \sum_{r=1}^{n} r b_r$$

then $\mathrm{mult}_{\mathrm{A}_n}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{A}_{n-1}}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \mathrm{V}_{\overline{\mu}_2})$, where $\overline{\mu}_1$, $\overline{\mu}_2$, and $\overline{\mu}$ are given by (4.3.1).

Example: $n = 5$, $i = j = 1$, $k = 2$, $\mu_1 = (3, 1, 3, 2, 1)$, $\mu_2 = (4, 1, 2, 3, 4)$, $\mu = (1, 1, 8, 3, 4)$, $\overline{\mu}_1 = (4, 3, 2, 1)$, $\overline{\mu}_2 = (5, 2, 3, 4)$, $\overline{\mu} = (1, 9, 3, 4)$; the multiplicity is 24.

This rule is much cleaner in GL_{n+1} coordinates.

Reduction rule: If $\mu_1 = (a_0, \ldots, a_n)$, $\mu_2 = (b_0, \ldots, b_n)$, and $\mu = (c_0, \cdots, c_n)$ are dominant GL_{n+1} weights (again with $\sum c_i = \sum a_i + \sum b_i$), and $0 \leqslant i, j, k \leqslant n$ such that $i + j = k$, then if $c_k = a_i + b_j$ we have $\mathrm{mult}_{\mathrm{GL}_{n+1}}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{GL}_n}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \mathrm{V}_{\overline{\mu}_2})$, where $\overline{\mu}_1$, $\overline{\mu}_2$, and $\overline{\mu}$ are obtained by deleting the entries a_i, b_j, and c_k from μ_1, μ_2, and μ respectively.

Example: $n = 6$, $i = 1$, $j = 2$, $k = 3$, $\mu_1 = (16, 13, 12, 9, 7, 3, 0)$, $\mu_2 = (21, 16, 13, 12, 9, 5, 0)$, $\mu = (29, 28, 27, 26, 13, 9, 4)$, $\overline{\mu}_1 = (16, 12, 9, 7, 3, 0)$, $\overline{\mu}_2 = (21, 16, 12, 9, 5, 0)$, $\overline{\mu} = (29, 28, 27, 13, 9, 4)$; the multiplicity is 108.

This GL_{n+1} rule appears as Reduction Formula I for Schubert calculus in [GH, p. 202]. (The rule given there does not appear exactly as stated above, but is equivalent to it after making the translation from intersecting three Schubert cycles to computing the multiplicity of a representation in a tensor product, and after using the indexing for the fundamental weights starting with zero.)

4.4. A three-factor reduction rule. The most important case for Littlewood-Richardson problems (i.e., the problem of computing $\mathrm{mult}_{\mathrm{G}}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k})$) is the case with two factors, as in the examples above. The main theorem, however, gives the construction of reduction rules for an arbitrary number of factors, and we give a three-factor example here. For simplicity, we just repeat the GL_{n+1} to GL_n reduction in §4.3, but now using the multiplication $h^i \cdot h^j \cdot h^k = h^m$ in $\mathrm{H}^*(\mathbf{P}^n, \mathbf{Z})$ whenever $0 \leqslant i, j, k, m \leqslant n$ and $m = i + j + k$. This gives:

Reduction rule: For any $0 \leqslant i, j, k, m \leqslant n$ with $i + j + k = m$, then for any dominant GL_{n+1} weights $\mu_1 = (a_0, \ldots, a_n)$, $\mu_2 = (b_0, \ldots, b_n)$, $\mu_3 = (c_0, \ldots, c_n)$, and $\mu = (d_0, \ldots, d_n)$, if $d_m = a_i + b_j + c_k$ then $\mathrm{mult}_{\mathrm{GL}_{n+1}}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2} \otimes \mathrm{V}_{\mu_3}) = \mathrm{mult}_{\mathrm{GL}_n}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \mathrm{V}_{\overline{\mu}_2} \otimes \mathrm{V}_{\overline{\mu}_3})$, where $\overline{\mu}_1$, $\overline{\mu}_2$, $\overline{\mu}_3$, and $\overline{\mu}$ are obtained by deleting the entries a_i, b_j, c_k and d_m from μ_1, μ_2, μ_3, and μ respectively. This rule generalizes to a larger number of factors in the obvious way.

Example: $n = 4$, $i = j = k = 1$, $m = 3$, $\mu_1 = (36, 28, 24, 16, 0)$, $\mu_2 = (40, 24, 20, 8, 0)$, $\mu_3 = (94, 14, 11, 9, 0)$, $\mu = (118, 68, 67, 66, 5)$, $\overline{\mu}_1 = (36, 24, 16, 0)$, $\overline{\mu}_2 = (40, 20, 8, 0)$, $\overline{\mu}_3 = (94, 11, 9, 0)$, $\overline{\mu} = (118, 68, 67, 5)$; the multiplicity is 196.

Even though the Littlewood-Richardson coefficients for the decomposition of the tensor product of two irreducible representations determine the coefficients for the decomposition of the tensor product of k irreducible representation, there does not seem to be an obvious argument for deducing the k-factor reduction rules from the two-factor reduction rules.

4.5. A codimension-two reduction. The previous examples have all been codimension-one reductions, i.e., starting with a codimension-one regular face of $\mathcal{C}(k)$ we obtain a rule with a single condition to check which reduces the rank of the group by one. In this section we give a codimension-two example. By Corollary 5.2.2 below, any codimension-r rule can be obtained as a succession of r codimension-one rules, but it is sometimes useful to be able to apply the rule "all at once". For instance, if n is the rank of G, than a codimension-n or - $(n-1)$ rule guarantees that the multiplicity of the corresponding component is one.

Suppose that G has type A_4. In order to avoid calculating in the cohomology ring of a two-step Grassmanian when working out the codimension-two reduction rule, we use a method explained in §4.8 below. Start with $w_1 = s_3s_4s_2$, $w_2 = s_4s_2s_3$, and $w = s_2s_3s_4s_2s_3s_2$, which have the property that $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$. Let $\mathrm{I} = \{\alpha_1, \alpha_2\}$. The elements $\widetilde{w}_1 = s_3s_4$, $\widetilde{w}_2 = s_4s_2s_3$, and $\widetilde{w} = s_2s_3s_4s_2s_3$ are the minimal representatives of w_1, w_2, and w in the corresponding cosets of $\mathcal{W}_{\mathrm{P}_\mathrm{I}}$, and therefore, as explained in §4.8, satisfy (2.6.1) with respect to I. For dominant weights $\mu_1 = (a_1, \ldots, a_4)$, $\mu_2 = (b_1, \ldots, b_4)$, and $\mu = (c_1, \ldots, c_4)$ we have

$$\begin{aligned}
\widetilde{w}_1^{-1}\mu_1 &= (a_1, a_2 + a_3, a_4, -a_3 - a_4) \\
\widetilde{w}_2^{-1}\mu_2 &= (b_1 + b_2, b_3 + b_4, -b_2 - b_3 - b_4, b_2 + b_3) \\
\widetilde{w}^{-1}\mu &= (c_1 + c_2 + c_3, c_4, -c_3 - c_4, -c_2).
\end{aligned}$$

The group $\overline{\mathrm{G}}$ is of type A_2, and restriction to $\overline{\mathrm{G}}$ ignores the last two coordinates in the expressions above, so

(4.5.1)
$$\begin{cases} \overline{\mu}_1 &= (a_1, a_2+a_3) \\ \overline{\mu}_2 &= (b_1+b_2, b_3+b_4) \\ \overline{\mu} &= (c_1+c_2+c_3, c_4). \end{cases}$$

The condition that $\overline{w}_1^{-1}\mu_1 + \overline{w}_2^{-1}\mu_2 - \overline{w}^{-1}\mu \in \mathrm{span}_{\mathbf{Q}}\,\mathrm{I}$ is given by the two linear conditions

(4.5.2)
$$\begin{cases} 2c_1 - c_2 - 4c_3 - 2c_4 &= (2a_1+4a_2+a_3+3a_4) + (2b_1 - b_2 + b_3 - 2b_4), \text{ and} \\ c_1 - 3c_2 - 2c_3 - c_4 &= (a_1 + 2a_2 - 2a_3 - a_4) + (b_1 + 2b_2 + 3b_3 - b_4). \end{cases}$$

This gives:

Reduction rule: If (4.5.2) holds, then $\mathrm{mult}_{\mathrm{A}_4}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1}\otimes \mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{A}_2}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1}\otimes \mathrm{V}_{\overline{\mu}_2})$, where $\overline{\mu}_1$, $\overline{\mu}_2$, and $\overline{\mu}$ are given by (4.5.1).

Example: $\mu_1 = (12,2,7,4)$, $\mu_2 = (3,6,4,15)$, $\mu = (22,1,1,7)$, $\overline{\mu}_1 = (12,9)$, $\overline{\mu}_2 = (9,19)$, $\overline{\mu} = (24,7)$; the multiplicity is 2.

4.6. A D_n to D_{n-1} reduction rule. Let G be of type D_n and let $\mathrm{I} = \{\alpha_2, \ldots, \alpha_n\}$. The quotient $\mathrm{Q}_n := \mathrm{G}/\mathrm{P}_\mathrm{I}$ is a smooth quadric hypersurface in $\mathbf{P}^{2n-1}$. The cohomology ring of Q_n is generated by h (the class of a hyperplane section) and two classes a and b of complex codimension $(n-1)$ (i.e., in the middle cohomology of Q_n) satisfying the relations

(4.6.1) $h^{n-1} = a+b, ha = hb, h^n a = 0, a^2 = b^2 = \frac{1}{2}(1-(-1)^n)[pt], ab = \frac{1}{2}(1+(-1)^n)[pt],$

where $[pt]$ indicates the class of a point. The cohomology ring of Q_n therefore has the presentation

$$\mathrm{H}^*(\mathrm{Q}_n, \mathbf{Z}) = \frac{\mathbf{Z}[h,a,b]}{\text{(relations in (4.6.1))}}.$$

The integral basis for $\mathrm{H}^*(\mathrm{Q}_n, \mathbf{Z})$ given by $\{h^k\}_{0\leqslant k\leqslant n-2}$ in codimension $\leqslant n-2$, a and b in codimension $n-1$, and $\{h^k a\}_{1\leqslant k \leqslant n-1}$ in codimensions n to $2(n-1)$ is a basis of Schubert classes in $\mathrm{H}^*(\mathrm{Q}_n, \mathbf{Z})$. We will only work out the most elementary example of a D_n to D_{n-1} reduction rule. If $k \leqslant n-2$ then h^k is the class of a Schubert cycle in $\mathrm{H}^*(\mathrm{Q}_n, \mathbf{Z})$ and the pullback to X is the class $[\Omega_{s_k s_{k-1}\cdots s_1}]$, as in the A_n case. For $k \leqslant n-3$ the action of $s_k \cdots s_1$ on dominant weights is also given by the same formula as in the A_n case.

For $0 \leqslant i,j,k \leqslant n-3$ with $k = i+j$, set $w_1 = s_i s_{i-1}\cdots s_1$, $w_2 = s_j s_{j-1}\cdots s_1$, and $w = s_k s_{k-1} \cdots s_1$. A short computation (which we omit) shows that $w_1^{-1}\cdot 0 + w_2^{-1}\cdot 0 - w^{-1}\cdot 0 \in \mathrm{span}_{\mathbf{Z}_{\geqslant 0}}\,\mathrm{I}$, and so w_1, w_2, and w satisfy (2.6.1) with respect to I.

For dominant weights $\mu_1 = (a_1, \cdots, a_n)$, $\mu_2 = (b_1, \ldots, b_n)$ and $\mu = (c_1, \cdots, c_n)$ the condition that $w_1^{-1}\mu_1 + w_2^{-1}\mu_2 - w^{-1}\mu \in \mathrm{span}_{\mathbf{Q}}\,\mathrm{I}$ is

$$\textbf{(4.6.2)}\quad 2\left(\sum_{r=k+1}^{n-2} c_r\right) + c_{n-1} + c_n = 2\left(\sum_{r=i+1}^{n-2} a_r\right) + a_{n-1} + a_n + 2\left(\sum_{r=j+1}^{n-2} b_r\right) + b_{n-1} + b_n.$$

Reduction rule: For any $0 \leqslant i, j, k \leqslant n-3$ with $k = i+j$, if μ_1, μ_2, and μ satisfy (4.6.2) then $\mathrm{mult}_{\mathrm{D}_n}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{D}_{n-1}}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \mathrm{V}_{\overline{\mu}_2})$ where $\overline{\mu}_1$, $\overline{\mu}_2$, and $\overline{\mu}$ are given by (4.3.1).

Example: $n = 5$, $i = j = 1$, $k = 2$, $\mu_1 = (7, 1, 6, 5, 7)$, $\mu_2 = (4, 1, 4, 3, 4)$, $\mu = (1, 1, 16, 4, 7)$, $\overline{\mu}_1 = (8, 6, 5, 7)$, $\overline{\mu}_2 = (5, 4, 3, 4)$, $\overline{\mu} = (1, 17, 4, 7)$; the multiplicity is 514.

In order to get a D_n to D_{n-1} rule where the reduction formulas are different from the A_n case, one only has to use deeper cohomology classes (e.g., multiplications involving a or b). Similar "A_n-like" formulas hold for C_n to C_{n-1} and B_n to B_{n-1} reductions if one uses low-codimension multiplications in $\mathrm{G}/\mathrm{P}_\mathrm{I}$ ($\mathrm{I} = \{\alpha_2, \ldots, \alpha_n\}$ as above), although the condition to check in order to apply the rule is different (e.g., compare (4.6.2) and (4.3.2)).

4.7. A C_n to A_{n-1} reduction. Let G be of type C_n and $\mathrm{I} = \{\alpha_1, \ldots, \alpha_{n-1}\}$. The quotient $\mathrm{LG}_n := \mathrm{G}/\mathrm{P}_\mathrm{I}$ is the *Lagrangian Grassmanian*, the Grassmanian of Lagrangian n-planes in a $2n$-dimensional complex vector space with a non-degenerate skew-symmetric form.

Similar to the ordinary Grassmanians, the Schubert basis for $\mathrm{H}^*(\mathrm{LG}_n, \mathbf{Z})$ is given by classes $\sigma_{a_1,a_2,\ldots,a_m}$ so that the corresponding partition $(a_1, a_2, \ldots, a_m)$ fits into an $n \times n$ box, but with the additional restriction that the partition be *strict*, i.e., that $a_1 > a_2 > \ldots > a_m \geqslant 1$ (see [FP, p. 29]). For any $a \geqslant 1$ set $u_a = s_{n+1-a}s_{n+2-a}\cdots s_{n-1}s_n$; then for any strict partition $n \geqslant a_1 > a_2 > \cdots > a_m \geqslant 1$ the pullback of the class $\sigma_{a_1,a_2,\ldots,a_m}$ to $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ is the class $[\Omega_w]$ with $w = u_{a_m}u_{a_{m-1}}\cdots u_{a_2}u_{a_1}$.

We will give only the simplest reduction rule, corresponding to the multiplication $\sigma_1 \cdot \sigma_2 = 2\sigma_3 + \sigma_{2,1}$ in cohomology. We must choose w to be $w = u_1u_2$ (i.e., so that $[\Sigma_w]$ is the pullback of $\sigma_{2,1}$) in order to satisfy (2.6.2)(*ii*). Setting $w_1 = s_n$, $w_2 = s_{n-1}s_n$, and $w = s_ns_{n-1}s_n$, then w_1, w_2, and w satisfy (2.6.1) with respect to I (condition (2.6.1)(*iii*) holds since $w_1 \cdot 0 + w_2 \cdot 0 - w \cdot 0 = 2\alpha_{n-1} \in \mathrm{span}_{\mathbf{Z}_{\geqslant 0}} \mathrm{I}$). The group we are reducing to is of type A_{n-1}, obtained by removing the last vertex of the Dynkin diagram for C_n.

If $\mu_1 = (a_1, \cdots, a_n)$, $\mu_2 = (b_1, \ldots, b_n)$ and $\mu = (c_1, \cdots, c_n)$ are dominant weights then

$$\begin{aligned} w_1^{-1}\mu_1 &= (a_1, a_2, \ldots, a_{n-3}, a_{n-2}, a_{n-1} + 2a_n, -a_n) \\ w_2^{-1}\mu_2 &= (b_1, b_2, \ldots, b_{n-3}, b_{n-2} + b_{n-1}, b_{n-1} + 2b_n, -b_{n-1} - b_n) \\ w^{-1}\mu &= (c_1, c_2, \ldots, c_{n-3}, c_{n-2} + c_{n-1} + 2c_n, c_{n-1}, -c_{n-1} - c_n). \end{aligned}$$

Restriction to $\overline{\mathrm{G}}$ ignores the last entry, so

$$\textbf{(4.7.1)}\qquad \begin{cases} \overline{\mu}_1 = (a_1, a_2, \ldots, a_{n-3}, a_{n-2}, a_{n-1} + 2a_n) \\ \overline{\mu}_2 = (b_1, b_2, \ldots, b_{n-3}, b_{n-2} + b_{n-1}, b_{n-1} + 2b_n) \\ \overline{\mu} = (c_1, c_2, \ldots, c_{n-3}, c_{n-2} + c_{n-1} + 2c_n, c_{n-1}). \end{cases}$$

The condition that $w_1^{-1}\mu_1 + w_2^{-1}\mu_2 - w^{-1}\mu$ lie in $\mathrm{span}_{\mathbf{Q}}\, \mathrm{I}$ is

(4.7.2)
$$\sum_{r=1}^{n} rc_r - 2c_{n-1} - 4c_n = \sum_{r=1}^{n} ra_r - 2a_n + \sum_{r=1}^{n} rb_r - 2b_{n-1} - 2b_n.$$

Reduction rule: If (4.7.2) holds then $\mathrm{mult}_{\mathrm{C}_n}(\mathrm{V}_\mu, \mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}) = \mathrm{mult}_{\mathrm{A}_{n-1}}(\mathrm{V}_{\overline{\mu}}, \mathrm{V}_{\overline{\mu}_1} \otimes \mathrm{V}_{\overline{\mu}_2})$ where $\overline{\mu}_1$, $\overline{\mu}_2$, and $\overline{\mu}$ are given by (4.7.1).

Example: $n = 5$, $\mu_1 = (8, 4, 3, 1, 3)$, $\mu_2 = (3, 2, 1, 6, 1)$, $\mu = (6, 6, 14, 1, 1)$, $\overline{\mu}_1 = (8, 4, 3, 7)$, $\overline{\mu}_2 = (3, 2, 7, 8)$, and $\overline{\mu} = (6, 6, 17, 1)$; the multiplicity is 31.

Remark on saturation. If $(\mu_1, \ldots, \mu_k, \mu)$ is an integral point of $\mathcal{C}(k)$ it does not necessarily imply that V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. The problem of determining the integral points for which this implication does hold is known as the saturation problem. For any integral point of $\mathcal{C}(k)$ it is known that the implication holds for some positive multiple of that point, and that the multiple can be bounded by a constant depending only on G. The cone $\mathcal{C}(k)$ (respectively a face F of $\mathcal{C}(k)$) is called *saturated* if the implication holds for every integral point in the cone (respectively on the face). In type A, all cones are saturated by the theorem of Knutson-Tao [KT, p. 1084]. If F is a regular face such that the corresponding reduction rule reduces to a group of type A, as in the example above, then the reduction theorem and the result of Knutson-Tao imply that F is saturated.

4.8. A rule for producing reduction rules. Suppose that $w_1, \ldots, w_k$, and w are elements of $\mathcal{W}$ such that

(4.8.1)
$$\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i},$$

i.e., Φ_w is the disjoint union of Φ_{w_1} through Φ_{w_k}. In the classical cases one can check that (4.8.1) implies that $\cap_{i=1}^{k}[\Omega_{w_i}] \cdot [\mathrm{X}_w] = 1$, and an argument proving this for general semisimple G will appear in [KP]. If I′ is the empty set (so $\mathrm{P}_{\mathrm{I}'} = \mathrm{B}$ and $\mathcal{W}_{\mathrm{P}_{\mathrm{I}'}} = \{e\}$) then $w_1, \ldots, w_k$, and w satisfy (2.6.1) with respect to I′ (condition (2.6.1)(*iii*) follows from (4.8.1) and (4.1.1)). Thus $w_1, \ldots, w_k$, and w describe a codimension-n regular face of $\mathcal{C}(k)$ and a corresponding codimension-n reduction rule, where n is the rank of G.

Furthermore, for any subset I of the simple roots, if we set $\widetilde{w}_1, \ldots, \widetilde{w}_k$, and $\widetilde{w}$ to be the shortest elements in the cosets $w_1\mathcal{W}_{\mathrm{P}_\mathrm{I}}, \ldots, w_k\mathcal{W}_{\mathrm{P}_\mathrm{I}}$, and $w\mathcal{W}_{\mathrm{P}_\mathrm{I}}$ respectively, then [DR1, Lemma 7.1.3] shows that $\widetilde{w}_1, \ldots, \widetilde{w}_k$ and $\widetilde{w}$ satisfy (2.6.1) with respect to I, yielding a codimension $n - |\mathrm{I}|$ face of $\mathcal{C}(k)$ and a corresponding reduction rule. I.e., the elements $w_1, \ldots, w_k$, and w give a family of reduction rules, one for each subset I of simple roots. I do not know if all regular faces arise via this procedure.

Any face containing a regular face is itself regular, and of course, the codimension $n - |\mathrm{I}|$ faces above are simply all the faces containing the codimension-n face corresponding to $w_1, \ldots, w_k$, and w. The question as to whether all regular faces arise via the procedure above is therefore equivalent to the question as to whether every regular face contains a regular codimension-n face.

5. FURTHER REMARKS

5.1. GIT interpretation of the reduction theorem. Suppose that F is a regular face of $\mathcal{C}(k)$, and let I, w_1, …, w_k, and w be the data parametrizing F given by Theorem 2.6.3. Let $\psi\colon \overline{\mathrm{X}}^{k+1} \longrightarrow \mathrm{X}^{k+1}$ be the embedding given in the reduction theorem. For any point $(\mu_1, \ldots, \mu_k, \mu)$ of F in the strictly dominant locus, the line bundle $\mathrm{L} := \mathrm{L}_{-w_0\mu_1} \boxtimes \cdots \boxtimes \mathrm{L}_{-w_0\mu_k} \boxtimes \mathrm{L}_{\mu}$ is very ample on X^{k+1}, and hence its pullback $\overline{\mathrm{L}} := \psi^*\mathrm{L}$ is very ample on $\overline{\mathrm{X}}^{k+1}$. For all $m \geqslant 0$ the reduction theorem implies that pullback by ψ induces an isomorphism $\psi^*\colon \mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \mathrm{L}^m)^{\mathrm{G}} \xrightarrow{\sim} \mathrm{H}^0(\overline{\mathrm{X}}^{k+1}, \overline{\mathrm{L}}^m)^{\overline{\mathrm{G}}}$.

The $\overline{\mathrm{G}}$-equivariant embedding ψ induces a map of GIT quotients $\overline{\mathrm{X}}^{k+1} /\!\!/_{\overline{\mathrm{L}}} \overline{\mathrm{G}} \longrightarrow \mathrm{X}^{k+1} /\!\!/_{\mathrm{L}} \mathrm{G}$, and the equality of pullbacks above for all $m \geqslant 0$ implies that this map is an isomorphism.

5.2. Reduction to $\mathcal{C}_{\overline{\mathrm{G}}}(k)$. If F is a regular face of $\mathcal{C}(k)$, and $\overline{\mathrm{G}}$ the corresponding group provided by the theorem, then reduction gives a map from F to $\mathcal{C}_{\overline{\mathrm{G}}}(k)$, the Littlewood-Richardson cone of $\overline{\mathrm{G}}$. In this section we prove some basic results about this reduction map.

Recall that for any polyhedral cone $\mathcal{C}$ in a vector space E every point $p \in \mathcal{C}$ lies on the relative interior of a unique face. The dimension of this face is the same as the dimension of the subspace of E spanned by the set $\{\varepsilon \in \mathrm{E} \mid p \pm \varepsilon \in \mathcal{C}\}$.

Proposition (5.2.1) — Suppose that F is a regular face of codimension r, that $p = (\mu_1, \ldots, \mu_k, \mu)$ is a point of F in the strictly dominant locus, and that p lies on the relative interior of a face of $\mathcal{C}(k)$ of codimension r'. Then the image of p under the reduction map $\mathrm{F} \longrightarrow \mathcal{C}_{\overline{\mathrm{G}}}(k)$ lies on the relative interior of a regular face of codimension $r' - r$.

Proof. Let $(\overline{\mu}_1, \mu_1')$, …, $(\overline{\mu}_k, \mu_k')$, and $(\overline{\mu}, \mu')$ be the restrictions of $w_1^{-1}\mu_1, \ldots w_k^{-1}\mu_k$, and $w^{-1}\mu$ respectively to $\bar{\mathfrak{t}}$ and $\mathfrak{a}$ under the splitting $\mathfrak{t} = \bar{\mathfrak{t}} \oplus \mathfrak{a}$ from §2.3, so that $\overline{p} := (\overline{\mu}_1, \ldots, \overline{\mu}_k, \overline{\mu})$ is the image of p under the reduction map. By the discussion at the end of §2.3, $\overline{p}$ is strictly dominant, and so we only need to check the statement on codimension. Write

$$p = \Big((\overline{\mu}_1, \mu_1'), \ldots, (\overline{\mu}_k, \mu_k'), (\overline{\mu}, \mu') \Big),$$

meaning that we have changed basis by w_i^{-1} (or w^{-1}) and applied the splitting to each entry. Let $\overline{\varepsilon} := (\overline{\varepsilon}_1, \ldots, \overline{\varepsilon}_k, \overline{\varepsilon}_{k+1})$ be a tuple with each $\overline{\varepsilon}_i \in \bar{\mathfrak{t}}^*$, $\varepsilon' = (\varepsilon_1', \ldots, \varepsilon_k', \varepsilon_{k+1}')$ be a tuple with each $\varepsilon_i' \in \mathfrak{a}^*$, and

$$p \pm (\overline{\varepsilon}, \varepsilon') := \Big((\overline{\mu}_1 \pm \overline{\varepsilon}_1, \mu_1' \pm \varepsilon_1'), \ldots, (\overline{\mu}_k \pm \overline{\varepsilon}_k, \mu_k' \pm \varepsilon_k'), (\overline{\mu} \pm \overline{\varepsilon}_{k+1}, \mu' \pm \varepsilon_{k+1}') \Big).$$

The vector space map underlying the reduction map sends $p \pm (\overline{\varepsilon}, \varepsilon')$ to $\overline{p} \pm \overline{\varepsilon}$. We want to find those $\overline{\varepsilon}$ such that $\overline{p} \pm \overline{\varepsilon} \in \mathcal{C}_{\overline{\mathrm{G}}}(k)$ which can be realized as the image of $(\overline{\varepsilon}, \varepsilon')$ such that $p \pm (\overline{\varepsilon}, \varepsilon') \in \mathcal{C}(k)$. We can make the following simplifying assumptions: (*i*) since both $\mathcal{C}(k)$ and $\mathcal{C}_{\overline{\mathrm{G}}}(k)$ are rational cones, we may restrict to rational $\overline{\varepsilon}$ and ε'. (*ii*) since it is only the dimension of the vector space spanned by $\overline{\varepsilon}$ (respectively $(\overline{\varepsilon}, \varepsilon')$) that matters, we may scale these vectors and assume they are arbitrarily small. In particular, since p is strictly dominant, we may (after scaling $(\overline{\varepsilon}, \varepsilon')$) assume that both $p \pm (\overline{\varepsilon}, \varepsilon')$ are dominant.

Since p lies on the face F, in order for $p \pm (\overline{\varepsilon}, \varepsilon')$ to be in $\mathcal{C}(k)$ a necessary condition is that $p \pm (\overline{\varepsilon}, \varepsilon')$ satisfy the linear conditions defining F. In these coordinates, the condition is simply that $\sum \varepsilon'_i = 0 \in \mathfrak{a}^*$. If this condition holds, (and since $p \pm (\overline{\varepsilon}, \varepsilon')$ are dominant) we may apply the reduction rule. Scaling by some positive integer m so that all weights are integral, the reduction rule says that

$$\mathrm{mult}_{\mathrm{G}}(\mathrm{V}_{m(\overline{\mu}\pm\overline{\varepsilon}_{k+1},\mu'\pm\varepsilon'_{k+1})}, \mathrm{V}_{m(\overline{\mu}_1\pm\overline{\varepsilon}_1,\mu'_1\pm\varepsilon'_1)}\otimes\cdots\otimes\mathrm{V}_{m(\overline{\mu}_k\pm\overline{\varepsilon}_k,\mu'_k\pm\varepsilon'_k)}) = \mathrm{mult}_{\overline{\mathrm{G}}}(\mathrm{V}_{m(\overline{\mu}\pm\overline{\varepsilon}_{k+1})}, \mathrm{V}_{m(\overline{\mu}_1\pm\overline{\varepsilon}_1)} \otimes \cdots \otimes \mathrm{V}_{m(\overline{\mu}_k\pm\overline{\varepsilon}_k,)}).$$

Thus (subject to the simplifying assumptions above), $p\pm(\overline{\varepsilon}, \varepsilon') \in \mathcal{C}(k)$ if and only if $\sum \varepsilon'_i = 0 \in \mathfrak{a}^*$ and $\overline{p} \pm \overline{\varepsilon} \in \mathcal{C}_{\overline{\mathrm{G}}}(k)$. In particular this shows that (up to scaling) all $\overline{\varepsilon}$ such that $\overline{p} \pm \overline{\varepsilon} \in \mathcal{C}_{\overline{\mathrm{G}}}(k)$ may be realized, and that the kernel of the map $(\overline{\varepsilon}, \varepsilon') \longrightarrow \overline{\varepsilon}$ has dimension $k \dim_{\mathbf{C}}(\mathfrak{a}^*) = kr$. Counting dimensions then gives the proposition. □

Here are some immediate corollaries. First, the proposition implies the result mentioned in §4.5.

Corollary (5.2.2) — The reduction rule corresponding to a regular face of codimension r can be obtained as a succession of r codimension-one reduction rules.

Proof. Suppose that F is a regular face of codimension r, then F is contained in a codimension 1 face F′ which must also be regular. Let $\overline{\mathrm{G}}'$ be the group corresponding to F′, then by Proposition 5.2.1 the image of F under the codimension-one reduction map $\mathrm{F}' \longrightarrow \mathcal{C}(k)_{\overline{\mathrm{G}}'}$ is a regular face of codimension $r - 1$. Continuing inductively we obtain a succession of r codimension-one reduction rules. What remains is to check that the composition of these rules is the same rule as the codimension-r rule obtained from the face F. We briefly sketch how to produce at least one factorization such that this holds.

Suppose that the face F is determined by the data I, $w_1,\ldots, w_k$, and w as in Theorem 2.6.3. Let $\alpha_j \in \mathrm{I}$ be any simple root, and P_j the parabolic obtained by inverting α_j. Let $\widetilde{w}_1,\ldots, \widetilde{w}_k$, and $\widetilde{w}$ be the minimal representatives in the cosets $w_1\mathcal{W}_{\mathrm{P}_j}, \ldots w_k\mathcal{W}_{\mathrm{P}_j}$, and $w\mathcal{W}_{\mathrm{P}_j}$ respectively, and let $u_1,\ldots, u_k$, and $u \in \mathcal{W}_{\mathrm{P}_j}$ be the unique elements such that $w_1 = \widetilde{w}_1 u_1,\ldots, w_k = \widetilde{w}_k u_k$, and $w = \widetilde{w} u$. Then similarly to the proof of [DR1, Lemma 7.1.3] one can check that $\widetilde{w}_1,\ldots, \widetilde{w}_k$, and $\widetilde{w}$ satisfy conditions (2.6.1) with respect to $\{\alpha_j\}$ and so define an codimension-one face F′. Furthermore, $u_1,\ldots, u_k$, and u satisfy (2.6.1) with respect to $\mathrm{I} \setminus \{\alpha_j\}$ in the group $\overline{\mathrm{G}}'$, and parametrize the regular face corresponding to the image of F in $\mathcal{C}(k)_{\overline{\mathrm{G}}'}$. The corresponding codimension-one reduction rule is computed in coordinates (as in the examples above) by writing $\widetilde{w}_1^{-1}\mu_1,\ldots, \widetilde{w}_k^{-1}\mu_k$, and $\widetilde{w}^{-1}\mu$ in the basis of fundamental weights and dropping the j-th coordinate. This is the same as writing $w_1^{-1}\mu_1,\ldots, w_k^{-1}\mu_k$, and $w^{-1}\mu$ in the basis of fundamental weights, dropping the j-th coordinate, and then applying $u_1,\ldots, u_k$, and u to the result. This shows that the composition of the codimension-one and codimension-$(r-1)$ rule is equal to the codimension-r rule, and by induction that the composition of the succession of r codimension-one rules is equal to the original codimension-r rule. $\square$

Second, by taking a point p in the relative interior of F we obtain:

Corollary (5.2.3) — The image of the reduction map is a full dimensional subcone of $\mathcal{C}_{\overline{\mathrm{G}}}(k)$.

This reduction map is not surjective in general, and it would be interesting to know how to characterize the image.

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