CURVE CLASSES ON RATIONALLY CONNECTED VARIETIES

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Abstract. This note proves every curve on a rationally connected variety is algebraically equivalent to a $\mathbb{Z}$-linear combination of rational curves.

1. Introduction

In [7] and [10], the following question is asked by Professor János Kollár and Professor Claire Voisin:

1.1. Question. *For a smooth projective rationally connected variety over $\mathbb{C}$ with dimension n, is every intergral Hodge $(n-1, n-1)$–class a $\mathbb{Z}$–linear combination of cohomology classes of rational curves?*

This question can be separated into two questions, as in [10]:

- For a smooth projective rationally connected variety over $\mathbb{C}$, is every intergral hodge $(n-1, n-1)$–class a $\mathbb{Z}$–linear combination of cohomology classes of curves?
- For a smooth projective rationally connected variety over $\mathbb{C}$, is every curve class a $\mathbb{Z}$–linear combination of cohomology classes of rational curves?

While generally unknown, the dimension 3 case of the first question is implied by the following result of Professor Claire Voisin:

1.2. Theorem. [9] *For a smooth projective 3-fold which is uniruled or Calabi-Yau, every integral Hodge $(2,2)$–class is a $\mathbb{Z}$–linear combination of cohomology classes of curves.*

A restricted form of the second question can be traced back to Fano and is believed among the comunity ([7]) to be raised by Professor V.A. Iskovski around 1970's.

1.3. Question. *Let X be a smooth projective Fano variety over $\mathbb{C}$, is every curve C in X homologous to a $\mathbb{Z}$-linear combination of homology classes of rational curves?*

We will solve the second question in this note, resulting in the following main theorem:

1.4. Theorem. *Let X be a smooth projective rationally connected variety over $\mathbb{C}$ then every curve on X is algebraically equivalent to a $\mathbb{Z}$-linear combination of rational curves.*

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The idea of proof is to first lift any irreducible curve C in X to $X\times\mathbb{P}^1$, and we regard the latter as a fibration over $\mathbb{P}^1$. By result of [5], there are very free rational curves which are horizontal with respect to the projection to $\mathbb{P}^1$, and free rational curves supported in the fibers $X\times\{p\in\mathbb{P}^1\}$, just adding enough of these curves to form a comb– which can be smoothed to a new horizontal curve $\hat{C}$ with $H^1(\hat{C},\mathcal{N}_{\hat{C}})=0$ and which is "flexible" in the sense of [5], namely, the map

$$\overline{\mathcal{M}}_{g,0}(X\times\mathbb{P}^1,\beta)\to\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)$$

as defined in [2] will be proper and surjective at the point represented by the curve $\hat{C}$, where g(resp β, d.) is the genus (resp cohomology class, degree over $\mathbb{P}^1$.) of $\hat{C}$–this map will contract non-stable components which emerges after projection to $\mathbb{P}^1$, all candidancies of which are either:

- rational curves with at most 2 marked points,
- (arithmetic) elliptic curves with no marked points.

We can exclude the second case of elliptic curves by recalling that deformations of $\hat{C}$ will always be connected.

Now the hurwitz scheme $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)$ is irreducible. So just degenerating the image of $\hat{C}$ to a sum of rational curves–by surjectivity and what we marked before about the non-stable components, we will get a sum of rational curves in $\overline{\mathcal{M}}_{g,0}(X\times\mathbb{P}^1,\beta)$ algebraically equivalent to $\hat{C}$. Simply fush forward this equivalence relation back to X, we will get the result.

Recalling that algebraic equivalence implies cohomological equivalence and combining theorem 1.2, we have:

1.5. Corollary. *For a smooth projective rationally connected 3-fold, every integral Hodge $(2,2)$-class is a $\mathbb{Z}$-linear combination of cohomology classes of rational curves.*

In an upcoming paper [8] of the author with Zhiyu Tian, we will try to explore further application of the "trivial product" trick in the proof.

The author would like to thank Professor János Kollár for his constant support and enlightening comments on this proof, to Professor Claire Voisin who pointed out that one can actually prove the rational equivalence rather than algebraic, also to Professor Burt Totaro who first introduced to the author the question for rationally connected 3-fold, and pointed out several unambiguities in first editions of this note, and the most thanks should be attributted to Zhiyu Tian, who taught the author story of [5], and knowledge about smoothing curves and moduli space of stable maps, without his help the author would never even dreamt of getting these results.

2. Preliminaries

2.6. Definition. *Let X be smooth projective variety over $\mathbb{C}$. It is rationally connected if there is a rational curve passing through 2 general points of X. By a free (resp. very free) curve in X we mean a rational curve $C\subset X$ with $T_X|_C$ non-negative (resp. ample). It is well-known that X to be rationally connected is equivalent to the existence of very free curves on X.*

2.7. Definition. *Let C be a connected nodal curve, X any variety, we call a map $f : C \to X$ a stable map if every component of C which is mapped to constant are either:*

- *A curve with arithmetic genus > 1*
- *A curve with arithmetic genus 1 with at least 1 nodal point.*
- *A curve with arithmetic genus 0 with at least 3 nodal points.*

It is well-known that we have good compactified moduli stack of all stable maps $f : C \to X$, $\overline{\mathcal{M}}_{g,0}(X, \beta)$ and β is the cohomology class of C in X, for reference see [4].

2.8. Definition. *Let k be an arbitrary field. A comb with n teeth over k is a projective curve with $n+1$ irreducible components $C_0, C_1, \ldots, C_n$ over $\bar{k}$ satisfying the following conditions:*

1.) *The curve C_0 is defined over k.*
2.) *The union $C_1 \cup \cdots \cup C_n$ is defined over k. (Each individual curve may not be defined over k.)*
3.) *The curves $C_1, \ldots, C_n$ are smooth rational curves disjoint from each other, and each of them meets C_0 transversely in a single smooth point of C_0 (which may not be defined over k).*

The curve C_0 is called the handle of the comb, and $C_1, \ldots, C_n$ are called the teeth. A rational comb is a comb whose handle is a smooth rational curve.

3. Proof of the Main Theorem

Let $\mathcal{Y}$ be a smooth projective variety with a morphism $\pi : \mathcal{Y} \to \mathbb{P}^1$ whose general fibers are rationally connected.

For a class $\beta \in H_2(\mathcal{Y}, \mathbb{Z})$ having intersection number d with a fiber of the map π. We have then a natural morphism as in [2]:

$$\varphi : \overline{\mathcal{M}}_{g,0}(\mathcal{Y}, \beta) \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$$

defined by composing a map $f : C \to \mathcal{Y}$ with π and collapsing components of C as necessary to make the composition $\pi \circ f$ stable.

3.9. Lemma. *For a stable map $f : C \to \mathcal{Y}$ which is non-constant, the components that are contracted under*

$$\varphi : \overline{\mathcal{M}}_{g,0}(\mathcal{Y}, \beta) \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$$

are all rational.

Proof. By Definition 2.7, the only possible non-stable components are:

- Smooth rational curve with at most 2 intersection points with other components of C
- A nodal rational curve or a smooth elliptic curve which is a connected component of the curve contracted at some step.

The third case can be excluded since C will always be connected after contraction. $\square$

3.10. Definition. *Let $f : C \to \mathcal{Y}$ be a stable map from a nodal curve C of genus g to $\mathcal{Y}$ with class $f_*[C] = \beta$. We say that f is flexible relative to π if the map $\varphi : \overline{\mathcal{M}}_{g,0}(\mathcal{Y}, \beta) \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ is dominant at the point $[f] \in \overline{\mathcal{M}}_{g,0}(\mathcal{Y}, \beta)$ and $\pi : C \to \mathbb{P}^1$ is flat.*

3.11. Proposition. *A flexible curve $f : C \to \mathcal{Y}$ can be degenerated to an effective sum of rational curves in $\mathcal{Y}$.*

Proof. It is a classical fact that the variety $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ has a unique irreducible component whose general member corresponds to a flat map $g : C \to \mathbb{P}^1$, see [3]. Since the map $\varphi : \overline{\mathcal{M}}_{g,0}(\mathcal{Y}, \beta) \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ is proper, and $f : C \to \mathcal{Y}$ is *flexible* then φ will be surjective on the component of $f : C \to \mathcal{Y}$. By Lemma 3.9 it is enough to find a degeneration of $C \to \mathbb{P}^1$ in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ as a sum of rational curves, which is elementary.

□

3.12. Lemma. [1] *Let X be a smooth projective variety of dimension at least* 3 *over an algebraically closed field. Let $D \subset X$ be a smooth irreducible curve and M a line bundle on D. Let $C \subset X$ be a very free rational curve intersecting D and let $\hat{C}$ be a family of rational curves on X parametrized by a neighborhood of $[C]$ in* $\mathrm{Hom}(\mathbb{P}^1, X)$.

Then there are curves $C_1, \ldots, C_p \in \hat{C}$ such that $D^* = D \cup C_1 \cup \cdots \cup C_p$ is a comb and satisfies the following conditions:

1.) *The sheaf $\mathcal{N}_{D^*}$ is generated by global sections.*
2.) *$H^1(D^*, \mathcal{N}_{D^*} \otimes M^*) = 0$, where M^* is the unique line bundle on D^* that extends M and has degree* 0 *on the C_i.*

Which leads to the following:

3.13. Lemma. *For any curve C in a rationally connected variety X, and any $m \gg 0$ very free curves $C_1, ..., C_m$, such that $C \cup C_1 \cup C_2 ... \cup C_m$ is a comb as in Definition 2.8, then: There is a sub-comb $C \cup C_{i_1} \cup C_{i_2} ... \cup C_{i_k}$, $k \leq m$ which can be deformed to an irreducible curve C' with $H^1(C', \mathcal{N}_{C'}) = 0$, where $\mathcal{N}_{C'}$ is the normal bundle of C'.*

3.14. Remark. We note that C can be highly singular in X, but let C' be the normalization of C, embed it as $C' \to \mathbb{P}^3$ and then project a small deformation of the diagonal map $C' \to X \times \mathbb{P}^3$ to X–we can get a deformation of C as a smooth sub–curve $C' \subset X$, then we can apply Lemma 3.12 to get Lemma 3.13.

Here we restate the *First Main Construction* in the article [5].

3.15. Theorem. *Assume there is a multisection $(B \subset \mathcal{Y}) \to \mathbb{P}^1$ which lies in the smooth locus of $\pi : \mathcal{Y} \to \mathbb{P}^1$, then there are rational curves C_i's such that $B \cup C_1 \cup ... \cup C_m$ can be smoothed to a flexible curve of $\mathcal{Y} \to \mathbb{P}^1$.*

Now we can prove our main theorem.

Proof. Take $\mathcal{Y} = X \times \mathbb{P}^1$, for any irreducible curve $C \subset X$, lift it to a curve C' in $X \times 0 \subset \mathcal{Y}$. Since $\mathcal{Y}$ is rationally connected, we can add enough free curves of $\mathcal{Y}$ which are horizontal with respect to the projection $\mathcal{Y} \to \mathbb{P}^1$, such that the *comb* can be deformed by Lemma 3.13 to a multisection M of the fibration $\pi : \mathcal{Y} \to \mathbb{P}^1$. Then by Theorem 3.15, since the trivial family has good reduction everywhere, we can add some other rational curves to M to be smoothed to a *flexible* curve, and then by Proposition 3.11, it can be degenerated to a sum of rational curves, so C' is algebraically equivalent to an integral sum of rational curves in $\mathcal{Y}$, the rest is simply pushing forward back to X.

□

3.16. Remark. As suggested by Professor János Kollár, one can prove the main theorem directly on X, by smoothing $C \cup C_1 \cup C_2 ... \cup C_m$ to a curve C' with $H^1(C', \mathcal{T}_X|_{C'}) = 0$ and then use the same argument above for the natural forgetful map

$$\overline{\mathcal{M}}_{g,0}(X, \beta) \to \overline{\mathcal{M}}_{g,0}$$

which is again proper and surjective–this will be discussed in [8].

3.17. Remark. As suggested by Professor Claire Voisin, based upon our result about algebraic equivalence, one can actually prove that all curves on X are rationally equivalent to a $\mathbb{Z}$–linear combination of rational curves, by using a construction of Professor János Kollár, see [8] for detail.

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