Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don’t spend time on a problem that you already know how to solve.


THE PROBLEMS

1. For $0 \leq m \leq k < n$, prove that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

2. Show that $\sum_{k=1}^{n} k(n\choose{k}) = n2^{n-1}$, and that $\sum_{k=0}^{n} \binom{n}{k+1} = \frac{2^{n+1}-1}{n+1}$.

3. For $n, k \geq 0$ and $p$ prime, prove that $\binom{p^n}{p^k} \equiv \binom{n}{k} \pmod{p}$.

4. (Putnam 2004, B2) Let $m$ and $n$ be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$ 

5. (Putnam 1992, B2) For nonnegative integers $n$ and $k$, define $Q(n, k)$ to be the coefficient of $x^k$ in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k - 2j}.$$ 

6. (Putnam 2000, B2) Prove that $\frac{\gcd(m, n)}{n} \binom{n}{m}$ is an integer for all pairs of integers $n \geq m \geq 1$.

7. (Putnam 1991, B4) Suppose that $p$ is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$ 

8. (Putnam 1996, A5) If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by $p^2$.

9. If $n \equiv 0 \pmod{6}$, then calculate the number of subsets of $[n]$ whose size is congruent to $r \pmod{3}$ for each $r \in \{0, 1, 2\}$. 

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SOME BASIC COMBINATORICS IDENTITIES

1. The number of subsets of \([n] = \{1, 2, \ldots, n\}\) is \(2^n\).

2. The number of permutations of \([n]\) is \(n!\).

3. The binomial coefficient \(\binom{n}{k}\) is defined to be the number of \(k\)-element subsets of \([n]\). Then \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\).

4. \(\binom{n}{k} = \binom{n}{n-k}, \binom{n}{k} = n\binom{n-1}{k-1}, \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\).

5. (Newton’s Binomial formula) \((1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k\).

6. (Vandermonde convolution formula) \(\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}\).

7. \(\sum_{k=0}^{n} \binom{n}{k} = 2^n, \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}\).

8. The number of choices of \(k\) objects from \(n\) with repetitions allowed and order not significant is equal to the number of ways of choosing \(n\) non-negative integers whose sum is \(k\).

9. The number of \(n\)-tuples of non-negative integers \(x_1, x_2, \ldots, x_n\) with \(x_1 + x_2 + \cdots + x_n = k\) is \(\binom{n+k-1}{n-1}\).

10. (De Moivre formula) \((\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)\).

11. If \(p\) is a prime number, and \(k\) the largest exponent so that \(p^k\) divides \(n!\), then

\[
k = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots
\]

Another way to compute \(k\) is to write \(n\) in base \(p\), i.e., write \(n = a_0 + a_1 p + a_2 p^2 + \cdots\) with \(0 \leq a_i \leq p - 1\). Then \(k = \frac{n - \sum a_i p^i}{p - 1}\).

12. (Lucas Theorem) Let \(p\) be a prime and let \(m = a_0 + a_1 p + \cdots + a_k p^k, n = b_0 + b_1 p + \cdots + b_k p^k\), where \(0 \leq a_i, b_i \leq p - 1\) for \(i = 0, 1, \ldots, k - 1\) (i.e., write \(m\) and \(n\) in base \(p\)). Then

\[
\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \pmod{p}
\]