Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS

1. (a) Prove that if $a, b, c > 0$, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

(b) If $a_1, \ldots, a_n > 0$, then

$$\sum_{i=1}^{n} \frac{a_i}{S - a_i} \geq \frac{n}{n-1}$$

where $S = \sum_{i=1}^{n} a_i$.

2. For $a, b, c > 0$, the following inequality holds:

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{a+c}} + \sqrt{\frac{c}{a+b}} > 2$$

3. (Putnam 1993, B1) Find the smallest positive integer n such that for every integer m, with $0 < m < 2007$ there exists an integer k for which

$$\frac{m}{2007} < \frac{k}{n} < \frac{m+1}{2008}$$

4. Let x_1,..., x_n be positive numbers such that

$$\frac{1}{x_1 + 2007} + \frac{1}{x_2 + 2007} + \cdots + \frac{1}{x_n + 2007} \geq \frac{1}{2007}.$$

Prove that $\dfrac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 2007$.

5. For any integer $n \geq 1$, show that the following inequality holds

$$\sqrt{1 + \sqrt{2 + \cdots + \sqrt{n}}} < 2.$$

Some Inequalities to Know

- **Baseball inequality**
  If a, b, c, and d are positive, with $\dfrac{a}{b} < \dfrac{c}{d}$ then $\dfrac{a}{b} < \dfrac{a+c}{b+d} < \dfrac{c}{d}$.

- (a) For any $x \in \mathbb{R}$, $x^2 \geq 0$.
  (b) For any $a, b \in \mathbb{R}$, $a^2 + b^2 \geq 2ab$.
  (c) If $a, b > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.
  (d) If $a, b, c > 0$, then $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$.
  (e) If $a, b, c \in \mathbb{R}$ then $a^2 + b^2 + c^2 \geq ab + bc + ca$ with equality iff $a = b = c$.
  (f) If $a, b, c \in \mathbb{R}$ then $a^4 + b^4 + c^4 \geq abc(a + b + c)$ with equality iff $a = b = c$.

- **HM-GM-AM-QM**
  If $a_1, \ldots, a_n > 0$ and $p \geq 1$, then
  $$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \ldots a_n} \leq \frac{a_1 + \cdots + a_n}{n} \leq \sqrt[p]{\frac{a_1^p + \cdots + a_n^p}{n}}.$$

- **Jensen**
  (a) If $f : \mathbb{R} \to \mathbb{R}$ is a concave up function, then
  $$f\left(\sum_{i=1}^{n} p_i a_i\right) \leq \sum_{i=1}^{n} p_i f(a_i)$$
  for all $a_1, \ldots, a_n \in \mathbb{R}$ and $p_1, \ldots, p_n \geq 0$ with $p_1 + \cdots + p_n = 1$.
  (b) If $f : \mathbb{R} \to \mathbb{R}$ is a concave up function, then
  $$f\left(\frac{a_1 + \cdots + a_n}{n}\right) \leq \frac{f(a_1) + \cdots + f(a_n)}{n}.$$
  (c) If f is concave down, the inequalities go the other way.

- **Cauchy-Schwarz**
  If $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$, then
  $$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$
  with equality iff $\frac{x_i}{y_i} = k$ for $1 \leq i \leq n$.