Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don’t spend time on a problem that you already know how to solve.


The Problems

1. (a) Prove that if $a, b, c > 0$, then
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

(b) If $a_1, \ldots, a_n > 0$, then
$$\sum_{i=1}^{n} \frac{a_i}{S - a_i} \geq \frac{n}{n-1}$$
where $S = \sum_{i=1}^{n} a_i$.

2. For $a, b, c > 0$, the following inequality holds:
$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{a+c}} + \sqrt{\frac{c}{a+b}} > 2$$

3. (Putnam 1993, B1) Find the smallest positive integer $n$ such that for every integer $m$, with $0 < m < 2007$ there exists an integer $k$ for which
$$\frac{m}{2007} < \frac{k}{n} < \frac{m+1}{2008}$$

4. Let $x_1, \ldots, x_n$ be positive numbers such that
$$\frac{1}{x_1 + 2007} + \frac{1}{x_2 + 2007} + \cdots + \frac{1}{x_n + 2007} \geq \frac{1}{2007}.$$ 
Prove that $\sqrt[n]{x_1 x_2 \cdots x_n} \geq 2007$.

5. For any integer $n \geq 1$, show that the following inequality holds
$$\sqrt[2]{1 + \sqrt{2} + \cdots + \sqrt{n}} < 2.$$
Some Inequalities to Know

- **Baseball inequality**
  If $a, b, c,$ and $d$ are positive, with $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

- (a) For any $x \in \mathbb{R}$, $x^2 \geq 0$.
  (b) For any $a, b \in \mathbb{R}$, $a^2 + b^2 \geq 2ab$.
  (c) If $a, b > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.
  (d) If $a, b, c > 0$, then $\frac{b+c}{b} + \frac{c+a}{a} + \frac{a+b}{c} \geq 6$.
  (e) If $a, b, c \in \mathbb{R}$ then $a^2 + b^2 + c^2 \geq ab + bc + ca$ with equality iff $a = b = c$.
  (f) If $a, b, c \in \mathbb{R}$ then $a^4 + b^4 + c^4 \geq abc(a + b + c)$ with equality iff $a = b = c$.

- **HM-GM-AM-QM**
  If $a_1, \ldots, a_n > 0$ and $p \geq 1$, then
  \[
  \frac{1}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n} \leq \sqrt[p]{\frac{a_1^p + \cdots + a_n^p}{n}}.
  \]

- **Jensen**
  (a) If $f : \mathbb{R} \to \mathbb{R}$ is a concave up function, then
  \[f \left( \frac{\sum_{i=1}^{n} p_i a_i}{n} \right) \leq \frac{\sum_{i=1}^{n} p_i f(a_i)}{n}\]
  for all $a_1, \ldots, a_n \in \mathbb{R}$ and $p_1, \ldots, p_n \geq 0$ with $p_1 + \cdots + p_n = 1$.
  (b) If $f : \mathbb{R} \to \mathbb{R}$ is a concave up function, then
  \[f \left( \frac{a_1 + \cdots + a_n}{n} \right) \leq \frac{f(a_1) + \cdots + f(a_n)}{n}\]
  (c) If $f$ is concave down, the inequalities go the other way.

- **Cauchy-Schwarz**
  If $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$, then
  \[\left( \sum_{i=1}^{n} x_i y_i \right)^2 \leq \left( \sum_{i=1}^{n} x_i^2 \right) \left( \sum_{i=1}^{n} y_i^2 \right)\]
  with equality iff $\frac{x_i}{y_i} = k$ for $1 \leq i \leq n$. 