Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don’t spend time on a problem that you already know how to solve.


THE PROBLEMS

1. (Putnam 2001, B1) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

2. (Putnam 2002, A1) Let $k$ be a fixed positive integer. The $n$-th derivative of $\frac{1}{x^k - 1}$ has the form $P_n(x)(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

3. (Putnam 2003, B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

4. (Putnam 1992, B4) Let $p(x)$ be a nonzero polynomial of degree less than 2007 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{2007} p(x)}{dx^{2007}} \frac{p(x)}{x^3 - x} = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

5. (Putnam 2002, A4) For each integer $m$, consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$

For what values of $m$ is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

6. (Putnam 2004, A4) Show that for any positive integer $n$, there is an integer $N$ such that the product $x_1x_2 \cdots x_n$ can be expressed identically in the form

$$x_1x_2 \cdots x_n = \sum_{i=1}^{N} c_i(a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n)^n$$

where the $c_i$ are rational numbers and each $a_{i,j}$ is one of the numbers $-1, 0, 1$.

7. (Putnam 1999, A2) Let $p(x)$ be a polynomial that is nonnegative for all real $x$. Prove that, for some $k$, there are polynomials $f_1(x), \ldots, f_k(x)$ such that $p(x) = \sum_{j=1}^{k} (f_j(x))^2$.