Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don’t spend time on a problem that you already know how to solve.


The Problems

1. (Putnam 1994, A4) Let \( A \) and \( B \) be \( 2 \times 2 \) matrices with integer entries such that \( A, A + B, A + 2B, A + 3B, \) and \( A + 4B \) are all invertible matrices whose inverses have integer entries. Show that \( A + 5B \) is invertible and that its inverse has integer entries.

2. (Putnam 1992, B5) Let \( D_n \) denote the value of the \( (n - 1) \times (n - 1) \) determinant

\[
\begin{vmatrix}
3 & 1 & 1 & 1 & \cdots & 1 \\
1 & 4 & 1 & 1 & \cdots & 1 \\
1 & 1 & 5 & 1 & \cdots & 1 \\
1 & 1 & 1 & 6 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & n + 1
\end{vmatrix}
\]

Is the set \( \{ D_n/n! \} \) bounded?

3. (Putnam 1988, B5) For positive integers \( n \), let \( M_n \) be the \( 2n + 1 \) by \( 2n + 1 \) skew-symmetric matrix for which each entry in the first \( n \) subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is \(-1\). Find, with proof, the rank of \( M_n \). As examples,

\[
M_1 = \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
M_2 = \begin{pmatrix}
0 & -1 & -1 & 1 & 1 \\
1 & 0 & -1 & -1 & 1 \\
1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 1 & 0 & -1 \\
-1 & -1 & 1 & 1 & 0
\end{pmatrix}
\]

4. (Putnam 1994, B4) For \( n \geq 1 \), let \( d_n \) be the greatest common divisor of the entries of \( A^n - I \), where

\[
A = \begin{pmatrix}
3 & 2 \\
4 & 3
\end{pmatrix}
\quad \text{and} \quad
I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Show that \( \lim_{n \to \infty} d_n = \infty \).

5. (Putnam 1986, A4) A transversal of an \( n \times n \) matrix \( A \) consists of \( n \) entries of \( A \), no two in the same row or column. Let \( f(n) \) be the number of \( n \times n \) matrices \( A \) satisfying the following two conditions
(a) Each entry \( \alpha_{i,j} \) of \( A \) is in the set \( \{1, 0, -1\} \).
(b) The sum of the \( n \) entries of a transversal is the same for all transversals of \( A \). An example of such a matrix \( A \) is
\[
A = \begin{pmatrix}
-1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]
Determine with proof a formula for \( f(n) \) of the form
\[
f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,
\]
where the \( a_i 's \) and \( b_i 's \) are rational numbers.

6. Let \( A \) and \( B \) be matrices of size \( 3 \times 2 \) and \( 2 \times 3 \) respectively. Suppose that their product \( AB \) is given by
\[
AB = \begin{pmatrix}
8 & 2 & -2 \\
2 & 5 & 4 \\
-2 & 4 & 5
\end{pmatrix}.
\]
Show that the product \( BA \) is given by
\[
BA = \begin{pmatrix}
9 & 0 \\
0 & 9
\end{pmatrix}.
\]

7. Let \( a, b, p_1, p_2, \ldots, p_n \) be real numbers with \( a \neq b \). Define the polynomial \( f(x) \) by
\[
f(x) = (p_1 - x)(p_2 - x) \cdots (p_n - x).
\]
Show that
\[
\left| \begin{array}{ccccccc}
p_1 & a & a & a & \cdots & a & a \\
b & p_2 & a & a & \cdots & a & a \\
b & b & p_3 & a & \cdots & a & a \\
b & b & b & p_4 & \cdots & a & a \\
: & : & : & : & \ddots & : & : \\
b & b & b & b & \cdots & p_{n-1} & a \\
b & b & b & b & \cdots & b & p_n
\end{array} \right| = \frac{bf(a) - af(b)}{b - a}.
\]

8. (Putnam 1985, B3) Let
\[
\begin{array}{ccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\
a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\
: & : & : & \ddots
\end{array}
\]
be a doubly infinite (i.e., infinite in both vertical and horizontal directions) array of positive integers, and suppose that each positive integer appears exactly eight times in the array. Prove that \( a_{m,n} > mn \) for some pair of positive integers \((m, n)\).