Tensor products of C*-algebras over Abelian subalgebras [†]

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§1: Introduction

Suppose A is a C*-algebra and C is a unital abelian C*-subalgebra which is isomorphic to a unital subalgebra of the centre of $M(A)$, the multiplier algebra of A. Letting $\Omega = \widehat{C}$, so that we may write $C = C(\Omega)$, we call (following Blanchard [7]), A a $C(\Omega)$-*algebra*. Suppose that B is another $C(\Omega)$-algebra, and we form $A \otimes_C B$, the algebraic tensor product of A with B over C as follows: $A \otimes B$ is the algebraic tensor product over $\mathbb{C}$, $I_C = \left\{ \sum_{i=1}^n (f_i \otimes 1 - 1 \otimes f_i)x \mid f_i \in C,\ x \in A \otimes B \right\}$ is the ideal in $A \otimes B$ generated by $\langle\ f \otimes 1 - 1 \otimes f \mid f \in C\ \rangle$ and $A \otimes_C B = A \otimes B / I_C$. $A \otimes_C B$ is an involutive algebra over $\mathbb{C}$, and we shall be interested in deciding when $A \otimes_C B$ is a pre-C*-algebra; that is, when is there a C*-norm on $A \otimes_C B$? There is a C*-seminorm, which we denote $\| \cdot \|_{\text{C-min}}$, which is minimal in the sense that it is dominated by any semi-norm whose kernel contains the kernel of $\| \cdot \|_{\text{C-min}}$. Moreover, if $A \otimes_C B$ has a C*-norm, then $\| \cdot \|_{\text{C-min}}$ is a C*-norm on $A \otimes_C B$. The problem is to decide when $\| \cdot \|_{\text{C-min}}$ is a norm. It was shown by Blanchard ([7, Proposition 3.1]) that when A and B are continuous fields and C is separable then $\| \cdot \|_{\text{C-min}}$ is a norm. In this paper we show that $\| \cdot \|_{\text{C-min}}$ is a norm when C is a von Neumann algebra, and then we examine some consequences.

This question of the existence of a C*-norm on $A \otimes_C B$ arose in work characterizing approximately inner automorphisms of von Neumann algebras [13]. In Section 3 we shall show that when M and N are von Neumann algebras and Z is a common unital von Neumann subalgebra of their centres, then $M \otimes_Z N$ has a minimal C*-norm (Corollary 3.4). In fact we show that this minimal norm has a spatial realization; if $M \subseteq \mathcal{L}(H)$ and $N \subseteq \mathcal{L}(K)$, then the norm $\| \cdot \|_{\text{Z-min}}$ is the

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operator norm on $M \otimes_Z N$ when it is represented on $H \otimes_Z K$, the relative tensor product of Sauvageot (Theorem 3.3). In Section 4 we introduce $\|\cdot\|_{\text{Z-bin}}$, an analogue of the norm $\|\cdot\|_{\text{bin}}$ introduced by Effros and Lance [11]; and the injectivity of M can be characterized by the identity $\|\cdot\|_{\text{Z-bin}} = \|\cdot\|_{\text{Z-min}}$ on $M \otimes_Z N$ for arbitrary N (Theorem 4.6).

When A and B are continuous fields over Ω, the semi-norm $\|\cdot\|_{\text{C-min}}$ is the supremum over Ω of the fibre-wise minimal norms. More generally the norm $\|\cdot\|_{\text{C-min}}$ can be exhibited by realizing the algebras A and B as upper-semicontinuous fields over Ω, (see [8], [14, p. 232], and [25] for definitions). Let us consider an example where $C = C(S^1) = \{f \in C[0,1] \mid f(0) = f(1)\}$ and $A = B$ is the C*-subalgebra of $\mathrm{L}^\infty[0,1]$ generated by C[0,1] and $\chi_{[0,\frac{1}{2}]}$. Thus A and B consist of functions on [0,1] which are continuous except possibly at $\frac{1}{2}$, where they have left and right limits, which may not be equal. For $\omega \in \Omega$

$$A_\omega = \begin{cases} \mathbb{C}_{0^+} \oplus \mathbb{C}_{0^-} & \omega = 0 \\ \mathbb{C}_w & \omega \neq 0, 1/2 \\ \mathbb{C}_{1/2^-} \oplus \mathbb{C}_{1/2^+} & \omega = 1/2. \end{cases}$$

Thus

$$(A \otimes_C B)_\omega = \begin{cases} \mathbb{C}_{++} \oplus \mathbb{C}_{+-} \oplus \mathbb{C}_{-+} \oplus \mathbb{C}_{--} & \omega = 0,\ 1/2 \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

Elements of $A \otimes_C B$ are continuous functions complex valued functions on $[0^+, \frac{1}{2}^-] \cup [\frac{1}{2}, 0^-]$, with arbitrary values at the four isolated points $\{\frac{1}{2}^{+-}, \frac{1}{2}^{-+}, 0^{+-}, 0^{-+}\}$. In this case $I_C = \{x \in A \otimes B \mid x(\omega) = 0 \text{ for } \omega \in \Omega\}$. So $\|x\|_{\text{C-min}} = \sup_{\omega \in \Omega} \|x(\omega)\|_{\min}$ is a norm on $A \otimes_C B$. This example can be extended to an arbitrary finite number of disconnections of the circle.

§2: Preliminaries

Let us begin by recalling from [11] some notation. Let $S(A \otimes B)$ be the state space of $A \otimes_{\max} B$, and $S(A \otimes_{\min} B) \subseteq S(A \otimes B)$ the state space of $A \otimes_{\min} B$. Suppose $C = C(\Omega)$ and A and B are $C(\Omega)$-algebras. For $\omega \in \Omega$ let C_ω be the maximal ideal corresponding to ω, i.e. C_ω is the ideal of functions vanishing at ω. Let $\varepsilon_\omega : C \to \mathbb{C}$

be evaluation at ω. Let $I_\omega = AC_\omega = \overline{AC_\omega}$, $A_\omega = A/I_\omega$ (with the quotient map given by $a \mapsto \pi_\omega(a) = a(\omega)$), and $S_\omega(A) := S(A_\omega) = \{\varphi \in S(A) | \varphi|_C = \varepsilon_\omega\}$ Let $\sigma_\omega : B \to B/I_\omega = B_\omega$ be the quotient map corresponding to $\pi_\omega : A \to A/I_\omega = A_\omega$ defined above. Now for $x \in A \otimes B$, $\|x\|_{\text{min}} = \sup_{(\omega,\omega')\in\Omega^2} \|\pi_\omega \otimes \sigma_{\omega'}(x)\|_{\text{min}}$. In order to get a semi-norm vanishing on I_C we restrict to the diagonal in Ω^2; the resulting semi-norm we call the C-min semi-norm: for $x \in A \otimes B$ let $\|x\|_{\text{C-min}} = \sup_{\omega\in\Omega} \|\pi_\omega \otimes \sigma_\omega(x)\|_{\text{min}}$.

Definition 2.1

Let $A \otimes_{\text{C-min}} B$ be the completion of $A \otimes B/I_{\text{C-min}}$ where $I_{\text{C-min}} = \{x \in A \otimes B \mid \|x\|_{\text{C-min}} = 0\}$.

Note that we have a natural inclusion of C into $M(A \otimes_{\text{C-min}} B)$ as a unital sub-algebra of the centre, and so we may form $(A \otimes_{\text{C-min}} B)_\omega$ for $\omega \in \Omega$. Let $S_\omega(A \otimes_{\text{C-min}} B) = S((A \otimes_{\text{C-min}} B)_\omega)$. The states in $S_\omega(A \otimes_{\text{C-min}} B)$ are exactly those whose restrictions to C are evaluation at ω. Thus $S(A_\omega \otimes_{\text{min}} B_\omega) \subseteq S_\omega(A \otimes_{\text{C-min}} B)$, i.e. the natural map $A \otimes_{\text{min}} B \to A_\omega \otimes_{\text{min}} B_\omega$ factors through $(A \otimes_{\text{C-min}} B)_\omega$.

That the semi-norm $\|\cdot\|_{\text{C-min}}$ has a minimal property was noticed by Elliott [12, Sublemma 2.1], (this proof can also be found in [7, 2.2]).

Lemma 2.2 ([12, Sublemma 2.1])

Let $\|\cdot\|_\alpha$ be a semi-norm on $A \otimes B$ such that for $x = a \otimes b \in A \otimes B$, $\|x\|_\alpha = 0$ implies that $\|x\|_{\text{C-min}} = 0$. Then for all $x \in A \otimes B$, $\|x\|_{\text{C-min}} \leq \|x\|_\alpha$; and the semi-norm $\|\cdot\|_{\text{C-min}}$ is minimal among all semi-norms on $A \otimes B$ whose kernel contains the kernel of $\|\cdot\|_{\text{C-min}}$.

Remark 2.3

It follows from Lemma 2.2 that if $\|\cdot\|_\alpha$ is a norm on $A \otimes_C B$, then $\|\cdot\|_{\text{C-min}}$ is also a norm.

It is shown in Blanchard [7, Proposition 3.1], that when A and B are continuous fields and C is separable, then $\|\cdot\|_{\text{C-min}}$ is a norm on $A \otimes_C B$. Outwith this case either result is possible: when A and B are not continuous fields there are examples where $\|\cdot\|_{\text{C-min}}$ is a norm on $A \otimes_C B$ and cases where it is not. Letting $A = B$ be the C*-algebra of L^∞[0,1] generated by C[0,1] and $\chi_{[0,\frac{1}{2}]}$, the characteristic function of the interval $[0,\frac{1}{2}]$, and $C = \{f \in A \mid f(0) = f(1)\}$ we have an example where A and B are not continuous fields but $\|\cdot\|_{\text{C-min}}$ is a norm on $A \otimes_C B$. On the other hand Blanchard [7] has given an example where A and B are not continuous fields and $\|\cdot\|_{\text{C-min}}$ is not a norm on $A \otimes_C B$.

§3: Tensor Products of Hilbert Spaces

Let M and N be von Neumann algebras and Z a common von Neumann subalgebra of their centres. We want to show that $\|\cdot\|_{\text{Z-min}}$ is in fact a norm on $M \otimes_{\mathrm{Z}} N$. It was observed by J. Glimm, [14, Lemma 10, p.233], that $\omega \mapsto \|m(w)\|$, for $m \in M$, is always continuous. For if we call this function $|m|_{\mathrm{Z}}$, then Glimm noted that $|m^*m|_{\mathrm{Z}} = \inf\{z \in Z \mid z \geq m^*m\} \in Z$ and is thus continuous. Hence M and N are continuous fields. Suppose $M \subseteq \mathcal{L}(H)$ and $N \subseteq \mathcal{L}(K)$; we shall show that there is a faithful representation of $M \otimes_{\mathrm{Z}} N$ on a Hilbert space $H \otimes_{\mathrm{Z}} K$ constructed by Sauvageot [27]. Let us recall his construction: [26, 27, 28]; but taking advantage the simplifications accruing from the use of an abelian algebra, Z. Instead of the spatial derivatives of Connes used by Sauvageot we shall use the notion of self-dual Hilbert C*-modules of Paschke [20] (as explained in [3, §2]). Let Z be an abelian von Neumann algebra acting on a Hilbert space H via a normal representation. For the purposes of taking tensor products we may suppose first that the representation is faithful and secondly that it is non-degenerate, i.e., the unit of Z gets sent to the identity operator on H. We shall denote by $\mathcal{L}_Z(H)$ the commutant of Z on H, and by $L^2(Z)$ the standard form of Z, (see Haagerup [15]). Let $X_H = \{T : L^2(Z) \to H \mid T$ intertwines the actions of $Z\}$. If $S, T, \in X_H$ then $S^*T \in \mathcal{L}_Z(L^2(Z)) = Z$. So we can define a Z-valued inner product on X_H by $(S \mid T) = S^*T$. X_H is thus a Hilbert Z-module; in fact it is self-dual (Paschke [20,§3 p. 449]) in that every bounded Z-linear map from X_H to Z is of the form $X_H \ni T \mapsto S^*T \in Z$ for some $S \in X_H$. To verify that X_H is self-dual one first shows that

$$\left\{ \sum_i S_i \xi_i \mid S_i \in X_H,\ \xi_i \in L^2(Z) \right\} \text{ is dense in } H, \tag{1}$$

this is proved in the second half of the proof of [3, Théorème 2.2]. Then given a bounded Z-linear map $\varphi : X_H \to Z$ one defines $T : H \to L^2(Z)$ by

$$T(\sum_i S_i \xi_i) = \sum_i \varphi(S_i)\xi_i. \tag{2}$$

By (1) this defines T on a dense subspace of H. The proof that (2) defines a bounded operator is on p. 79 of [24, proof of Theorem 6.5]. Then $T^* \in X_H$ and $TS = \varphi(S)$, so X_H is self-dual.

We can now construct $H \otimes_{\mathrm{Z}} K$ using the inducing construction of Rieffel [23, Theorem 5.9]. Give $X_H \otimes K$ the bilinear form $(S \otimes \xi \mid T \otimes \eta) = (T^* S\xi \mid \eta)$, (regarding T^*S as an operator in Z, and thus acting on K). Then mod out by the null-vectors and complete to obtain $X_H \otimes_{\mathrm{Z}} K$ (*c.f.* [4, 13.5.1]), which we shall denote by $H \otimes_{\mathrm{Z}} K$.

Recall that $\mathcal{L}_Z(H)$ denotes the commutant of Z on H. X_H is a left $\mathcal{L}_Z(H)$-module; for if $T \in \mathcal{L}_Z(H)$ and $S \in X_H$ then $TS \in X_H$. Thus for $T \in \mathcal{L}_Z(H)$ we have an operator $T \otimes_{\mathrm{Z}} 1_K \in \mathcal{L}_Z(H \otimes_{\mathrm{Z}} K)$. Moreover the map $\mathcal{L}_Z(H) \ni T \mapsto T \otimes_{\mathrm{Z}} 1_K \in \mathcal{L}_Z(H \otimes_{\mathrm{Z}} K)$ is normal. For if $\sum_i S_i \otimes \xi_i \in X_H \otimes K$, then $T \mapsto (T \otimes_{\mathrm{Z}} 1_K(\sum_i S_i \otimes \xi_i) \mid \sum_i S_i \otimes \xi_i) = \sum_{i,j}(S_j^* T S_i \xi_i \mid \xi_j)$ is normal on $\mathcal{L}_Z(H)$. Moreover $T \mapsto T \otimes_{\mathrm{Z}} 1_K$ is faithful; if $T \otimes_{\mathrm{Z}} 1_K = 0$ then $(T \sum_i S_i\xi_i | \sum_i S_i \xi_i) = 0$ for all $\sum_i S_i \xi_i \in H$. By (1) T is zero on a dense set; hence $T = 0$.

Similarly if $T \in \mathcal{L}_Z(K)$ we may define $1_H \otimes_{\mathrm{Z}} T \in \mathcal{L}_Z(H \otimes_{\mathrm{Z}} K)$ by $1_H \otimes_{\mathrm{Z}} T(\sum_i S_i \otimes \xi_i) = \sum_i S_i \otimes T\xi_i$. The inequality

$$\begin{aligned} \|1_H \otimes_{\mathrm{Z}} T(\sum_i S_i \otimes \xi_i)\|^2 &= \| \sum_i S_i \otimes T\xi_i\|^2 \\ &= \sum_{i,j} (S_j^* S_i T\xi_i \mid T\xi_i) = \sum_{i,j} (T^*T S_j^* S_i \xi_i \mid \xi_j) \\ &\leq \|T\|^2 \| \sum_i S_i \otimes \xi_i\|^2, \end{aligned}$$

shows that $1_H \otimes_{\mathrm{Z}} T$ is well-defined and bounded. And a computation similar to the one above shows that $\mathcal{L}_Z(K) \ni T \mapsto 1_H \otimes_{\mathrm{Z}} T \in \mathcal{L}_Z(H \otimes_{\mathrm{Z}} K)$ is a faithful normal $*$-homomorphism. So on $H \otimes_{\mathrm{Z}} K$ we have two von Neumann algebras: $\mathcal{L}_Z(H) \otimes_{\mathrm{Z}} 1_K$ and $1_H \otimes_{\mathrm{Z}} \mathcal{L}_Z(K)$; and now we want to verify (as in Sauvageot [27, Proposition 3.3]) that

$$(\mathcal{L}_Z(H) \otimes_{\mathrm{Z}} 1_K)' = 1_H \otimes_{\mathrm{Z}} \mathcal{L}_Z(K). \tag{3}$$

The idea is to use the amplification-reduction decomposition of H as a Z-module (see Dixmier [10, I.4.4 Theorem 3]): there is a Hilbert space $\widetilde{H}$ and a projection $e \in \mathcal{L}(\widetilde{H} \otimes L^2(Z))$ commuting with the amplified action $1_{\widetilde{H}} \otimes Z$ of Z, such that $H \simeq e(\widetilde{H} \otimes L^2(Z))$ via a unitary intertwining the two actions of Z.

First observe that if $H = L^2(Z)$ then $X_H = Z$ as a Hilbert Z-module, so $H \otimes_{\mathrm{Z}} K \simeq K$. In this case $\mathcal{L}_Z(H)$ is just Z (as Z is maximal abelian on $L^2(Z)$). Under this identification of $H \otimes_{\mathrm{Z}} K$ with K, $Z \otimes 1_K$ gets sent to the algebra $Z \subseteq \mathcal{L}_Z(K)$. So in this case we indeed have $(\mathcal{L}_Z(H) \otimes_{\mathrm{Z}} 1_K)' = Z' = \mathcal{L}_Z(K) = 1_H \otimes_{\mathrm{Z}} \mathcal{L}_Z(K)$.

Next observe that if $\widetilde{H}$ is a Hilbert space and we give $\widetilde{H} \otimes H$ the amplified Z-action $z \mapsto 1_{\widetilde{H}} \otimes z$ then

$$X_{\widetilde{H} \otimes H} \simeq \sum\nolimits^{\oplus}_{\dim \widetilde{H}} X_H = \left\{ (T_\nu) \mid \sum_\nu T_\nu^* T_\nu < \infty \text{ in } Z \right\},$$

i.e., the direct sum in the category of self-dual Hilbert W*-modules (see Paschke [20, pp. 457-458]).

Hence

$$X_{\widetilde{H} \otimes H} \otimes_{\mathrm{Z}} K \simeq \sum\nolimits^{\oplus}_{\dim \widetilde{H}} (X_H \otimes_{\mathrm{Z}} K),$$

where now the direct sum means the usual direct sum of Hilbert spaces. Thus we have the isomorphism

$$(\widetilde{H} \otimes H) \otimes_{\mathrm{Z}} K \simeq \widetilde{H} \otimes (H \otimes_{\mathrm{Z}} K).$$

So in particular when $H = L^2(Z)$ we have

$$(\widetilde{H} \otimes L^2(Z)) \otimes_{\mathrm{Z}} K \simeq \widetilde{H} \otimes K. \tag{4}$$

Since $\mathcal{L}_Z(\widetilde{H} \otimes L^2(Z)) = (1_{\widetilde{H}} \otimes Z)' = \mathcal{L}(\widetilde{H}) \mathbin{\overline{\otimes}} Z$, (where $\overline{\otimes}$ denotes the von Neumann tensor product of von Neumann algebras), we have, under the isomorphism (4),

$$\mathcal{L}_Z(\widetilde{H} \otimes L^2(Z)) \otimes_{\mathrm{Z}} 1_K \simeq \mathcal{L}(\widetilde{H}) \mathbin{\overline{\otimes}} Z \tag{5.}$$

On the other side we have, again under (4)

$$1_{\widetilde{H} \otimes L^2(Z)} \otimes_{\mathrm{Z}} \mathcal{L}_Z(K) \simeq 1_{\widetilde{H}} \otimes \mathcal{L}_Z(K). \tag{6}$$

Since the right hand sides of (5) and (6) are each other commutants we thus have

$$\Big(\mathcal{L}_Z(\widetilde{H} \otimes L^2(Z)) \otimes_{\mathrm{Z}} 1_K \Big)' = 1_{\widetilde{H} \otimes L^2(Z)} \otimes_{\mathrm{Z}} \mathcal{L}_Z(K). \tag{7}$$

Finally we must check what happens when we cut by a projection $e \in \mathcal{L}_Z(\widetilde{H} \otimes H)$. First observe that if $e \in \mathcal{L}_Z(H)$ then

$$X_{eH} = eX_H.$$

Let $e' \in \mathcal{L}(\widetilde{H}) \mathbin{\overline{\otimes}} Z$ be the image of $e \otimes_{\mathrm{Z}} 1_K$ under the isomorphism (4). So $H \otimes_{\mathrm{Z}} K = e \otimes_{\mathrm{Z}} 1_K((\widetilde{H} \otimes L^2(Z)) \otimes_{\mathrm{Z}} K) \simeq e'(\widetilde{H} \otimes K)$. Thus

$$\begin{aligned} \mathcal{L}_Z(H) \otimes_{\mathrm{Z}} 1_K &= e \otimes_{\mathrm{Z}} 1_K \cdot \Big[\mathcal{L}_Z(\widetilde{H} \otimes L^2(Z)) \otimes_{\mathrm{Z}} 1_K\Big] \cdot e \otimes_{\mathrm{Z}} 1_K \\ &\simeq e'[\mathcal{L}(\widetilde{H}) \mathbin{\overline{\otimes}} Z]e', \end{aligned} \tag{8}$$

and

$$\begin{aligned} 1_H \otimes_{\mathrm{Z}} \mathcal{L}_Z(K) &= e \otimes_{\mathrm{Z}} 1_K \cdot \Big[1_{\widetilde{H} \otimes L^2(Z)} \otimes_{\mathrm{Z}} \mathcal{L}_Z(K)\Big] \cdot e \otimes_{\mathrm{Z}} 1_K \\ &\simeq e'(1_{\widetilde{H}} \otimes \mathcal{L}_Z(K))e'. \end{aligned} \tag{9}$$

So using the rule: $(eMe)' = eM'e$ for a von Neumann algebra M and a projection $e \in M'$, we have, by applying this rule to the right hand sides of (8) and (9), that (3) holds in the general case when $H = e(\widetilde{H} \otimes L^2(Z))$.

Let us now recall a result of Murray and von Neumann [19, Theorem 3 p. 140] also [16, Theorem 5.5.4]. Let M be a von Neumann algebra on H with commutant M'. If $x_1, \ldots, x_n \in M$ and $y_1, \ldots, y_n \in M'$ and $\sum_i x_i y_i = 0$ then there is a projection $e = (e_{ij}) \in M_n(Z)$ such that $\sum_i x_i e_{ij} = 0$ for all j, and $\sum_j e_{ij} y_j = y_i$ for all i. Of course this says that $M \otimes_{\mathrm{Z}} M'$ is isomorphic to $\mathbb{C}(M, M')$, the $*$-subalgebra of $\mathcal{L}(H)$ generated by M and M', via the map $\sum_i x_i \otimes_{\mathrm{Z}} y_i \mapsto \sum_i x_i y_i$.

Theorem 3.1

Let Z be an abelian von Neumann algebra and suppose, as above, that we have faithful normal unital representations of Z on Hilbert spaces H and K. If $A \subseteq \mathcal{L}_Z(H)$ and $B \subseteq \mathcal{L}_Z(K)$ with $\overline{ZA} = A$ and $\overline{ZB} = B$, then we have an algebraic $*$-isomorphism $A \otimes_{\mathrm{Z}} B \to \mathbb{C}(A \otimes_{\mathrm{Z}} 1_K, 1_H \otimes_{\mathrm{Z}} B) \subseteq \mathcal{L}_Z(H \otimes_{\mathrm{Z}} K)$.

Proof: Let $M = \mathcal{L}_Z(H) = \mathcal{L}_Z(H) \otimes_{\mathrm{Z}} 1_K \subseteq \mathcal{L}(H \otimes_{\mathrm{Z}} K)$. Then $M' = 1_H \otimes_{\mathrm{Z}} \mathcal{L}_Z(K) = \mathcal{L}_Z(K)$.

For the purposes of this demonstration let us adopt the following notation. If $a \in A$, let $\tilde{a}$ be the operator $a \otimes_{\mathrm{Z}} 1_K$ on $H \otimes_{\mathrm{Z}} K$, similarly for $b \in B$, let $\tilde{b}$ be the operator $1_H \otimes_{\mathrm{Z}} b$ on $H \otimes_{\mathrm{Z}} K$. So suppose that $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$ and $\sum_i \tilde{a}_i \tilde{b}_i = 0$ (as an operator on $H \otimes_{\mathrm{Z}} K$).We must show that $\sum_i a_i \otimes b_i \in I_Z$, (c.f. the first paragraph of section 1). By the lemma of Murray and von Neumann there is a projection $e = (e_{ij})$ such that $\sum_i a_i e_{ij} = 0$ for all j and $\sum_j e_{ij} b_j = b_i$ for all i. Thus

$$\begin{aligned}
\sum_i a_i \otimes b_i &= \sum_i a_i \otimes (\sum_j e_{ij} b_j) \\
&= \sum_{ij} a_i \otimes e_{ij} b_j \\
&= \sum_{ij} a_i e_{ij} \otimes b_j + [1 \otimes e_{ij} - e_{ij} \otimes 1](a_i \otimes b_j) \\
&= \sum_j (\sum_i a_i e_{ij}) \otimes b_j + \sum_{ij} [1 \otimes e_{ij} - e_{ij} \otimes 1](a_i \otimes b_j) \\
&= \sum_{ij} [1 \otimes e_{ij} - e_{ij} \otimes 1](a_i \otimes b_j) \in I_Z.
\end{aligned}$$

Remark 3.2

Thanks to to this isomorphism we may write, without ambiguity, $\sum_i a_i \otimes_{\mathrm{Z}} b_i$ to mean either an element of $A \otimes_{\mathrm{Z}} B$ or an operator on $H \otimes_{\mathrm{Z}} K$.

Theorem 3.3

Suppose A, B, and Z are in Theorem 3.1, and for $x \in A \otimes B$, $\|x\|_{H \otimes_{\mathrm{Z}} K}$ denotes the norm of x as an operator on $H \otimes_{\mathrm{Z}} K$. Then $\|x\|_{\text{Z-min}} = \|x\|_{H \otimes_{\mathrm{Z}} K}$.

Proof: By Theorem 3.1 $\|\cdot\|_{H \otimes_{\mathrm{Z}} K}$ is a norm on $A \otimes_{\mathrm{Z}} B$ and by Elliott's generalization of Takesaki's theorem (Lemma 2.2), $\|x\|_{\text{Z-min}} \leq \|x\|_{H \otimes_{\mathrm{Z}} K}$ for all $x \in A \otimes_{\mathrm{Z}} B$. We must now prove the reverse inequality. Thus we must show that for each pair of vectors ξ and η in $H \otimes_{\mathrm{Z}} K$ the corresponding linear functional $\omega_{\xi,\eta}$ on $A \otimes B$ ($\omega_{\xi,\eta}(x) = (x\xi|\eta)$), is bounded with respect to the Z-min norm. In fact by the density of $X_K \cdot L^2(Z)$ in K we may take our ξ and η to be of the special form $\xi = S_1 \otimes S_2 \mu$ and $\eta = T_1 \otimes T_2 \nu$; for linear combinations of vectors of this form are dense in $X_H \otimes_{\mathrm{Z}} K$. On $A \otimes B$ the form acts as follows:

$$a \otimes b \mapsto ((a \otimes b)(S_1 \otimes S_2 \mu)|T_1 \otimes T_2 \nu) = \varphi(S_2^* a S_1 T_2^* b T_2),$$

where $\varphi = \omega_{\mu,\nu}$. To show that a linear form such as this is $\|\cdot\|_{\text{Z-min}}$-bounded, it suffices to suppose that φ is positive, as each $\omega_{\mu,\nu}$ is a linear combination of positive φ's, and finally we may suppose, by the Krein-Milman theorem, that φ is a pure state on Z. But if φ is pure then our linear form is such that

$$a \otimes b \mapsto \varphi(S_2^* a S_2)\varphi(T_2^* b T_1).$$

Such a form is in $A^* \times B^*$ and thus is always bounded in the min-norm. Since it also vanishes on I_Z, it is bounded in the Z-min norm.

Corollary 3.4

Let $M \subseteq \mathcal{L}(H)$ and $N \subseteq \mathcal{L}(K)$ be von Neumann algebras and Z a common sub-algebra of their centres. Then $\|\cdot\|_{\text{Z-min}} = \|\cdot\|_{H\otimes_{\mathrm{Z}}K}$ on $M \otimes_{\mathrm{Z}} N$.

Remark 3.5

We have a map $M \otimes_{\text{Z-min}} N \to \mathcal{L}(H \otimes_{\mathrm{Z}} K)$, whose kernel contains I_Z. It seems possible that this kernel is exactly the closure of I_Z in the norm $\|\cdot\|_{\text{Z-min}}$; but the only case in which we know that this holds is when either M or N is injective.

§4: Amenable inclusions of abelian subalgebras

Let M and N be von Neumann algebras and Z a common subalgebra of their centres. In section 3 we showed that there is a norm $\|\cdot\|_{\text{Z-min}}$ on $M \otimes_{\mathrm{Z}} N$ which is spatially implemented in that if $M \subseteq \mathcal{L}(H)$ and $N \subseteq \mathcal{L}(K)$ then there is a Hilbert space $H \otimes_{\mathrm{Z}} K$ and a faithful representation π_Z of $M \otimes_{\mathrm{Z}} N$ on $H \otimes_{\mathrm{Z}} K$ such that $\|\pi_Z(x)\| = \|x\|_{\text{Z-min}}$ for $x \in M \otimes_{\mathrm{Z}} N$. On $M \otimes_{\mathrm{Z}} N$ on can place the maximal norm $\|\cdot\|_{\text{Z-bin}}$ which is the analogue of the norm $\|\cdot\|_{\text{bin}}$ of Effros and Lance [11, p.10]. Recall that this norm can be constructed as follows. Given π and σ normal representation of M and N respectively on a Hilbert space H, such that $\pi(N) \subseteq \sigma(N)'$, then $\pi \otimes \sigma$ gives a representation of $M \otimes N$ on H via $\pi \otimes \sigma(\Sigma_i m_i \otimes n_i) = \Sigma_i \pi(m_i)\sigma(n_i)$. Then $\|x\|_{\text{bin}}$=sup $\{\|\pi \otimes \sigma(x)\| \mid (\pi, \sigma)$ is a commuting pair of normal representations such that $\pi(z) = \sigma(z)$ for z in $Z\}$.

The point of this section is to show that the equivalent ways of defining injectivity for a von Neumann algebra have analogues in the category of tensor products over Z. Suppose M acts standardly on H i.e. there are J and P such that (M, H, J, P) is a standard form in the sense of Haagerup [15] and $\mathbb{C}(M, M')$ is the $*$-algebra on H generated by M and M'. Let $\mathrm{C}^*(M, M')$ be the norm closure of $\mathbb{C}(M, M')$. In [13, Theorems 1.5 and 1.6] it was shown that if M is injective and semifinite then $\mathrm{C}^*(M, M') \simeq M \otimes_{\text{Z-min}} M'$, via the extension to $\mathrm{C}^*(M, M')$ of the natural isomorphism $\rho : \mathbb{C}(M, M') \to M \otimes_{\mathrm{Z}} M'$. In the converse direction, it follows from [11, Proposition 4.5] that if ρ extends to a (bounded) isomorphism $\mathrm{C}^*(M, M') \to M \otimes_{\text{Z-min}} M'$, then

$$M \otimes_{\min} M' \xrightarrow{\pi} M \otimes_{\text{Z-min}} M' \xrightarrow{\rho^{-1}} \mathrm{C}^*(M, M')$$

is the bounded extension of the map $\Sigma_i m_i \otimes m_i' \to \Sigma_i m_i m_i'$ and hence (by [11, Proposition 4.5]) M is semi-discrete and thus injective.

We refer the reader to section 1 of [13], section 1 of [1], and section 1 of [2] for the definitions and notations of correspondences between von Neumann algebras. Associated to an inclusion of von Neumann algebras $N \subseteq M$ is a correspondence from M to M (denoted Y_N in [1, p.339]). If the identity M-M correspondence is weakly contained in Y_N then we say that the inclusion $N \subseteq M$ is *amenable* (see [1, Definition 3.1] which was in turn based on [22, Definition 3.2.1]). If there is a normal conditional expectation $\mathcal{E}$ of M onto N then this correspondence Y_N, is the same as the M-M correspondence associated to the completely positive map $\mathcal{E}$: $M \to M$, (see [3, Théorème 2.14]).

In what follows Z will denote a von Neumann subalgebra of the centre of M. Given any normal state φ on M there is a condition expectation, $\mathcal{E}$, of M onto Z along φ; i.e. $\varphi \circ \mathcal{E} = \varphi$. This is a special case of a more general result of Takesaki [29, Theorem, p.309] (a proof taking advantage of the simplifications arising from having the subalgebra abelian can be found in [18, Lemma 1.1]). Thus the correspondence Y_Z arising from the inclusion $Z \subseteq M$ is what was denoted $H_{\mathcal{E}}$ in [13, § 1] and $H \otimes_Z H$ in section 3 (letting H be a space upon which M acts standardly). Hence $\pi_{\mathcal{E}}$, the representation of $M \otimes M'$ on $H_{\mathcal{E}}$ associated to this correspondence, realizes the norm $\| \cdot \|_{\text{Z-min}}$. By the minimality of $\| \cdot \|_{\text{Z-min}}$ we have that the map $\rho : \mathbb{C}(M, M') \to M \otimes_{\text{Z-min}} M'$ is bounded and thus extends to a surjective homomorphism

$$\rho : \mathrm{C}^*(M, M') \to M \otimes_{\text{Z-min}} M'.$$

In [13, Theorem 1.5] this was proved under the hypothesis that M is semi-finite, using a different technique. Since $M \otimes_{\text{Z-min}} M'$ is a quotient of $M \otimes_{\min} M'$ it follows from [11, Proposition 4.5] that if ρ is an isomorphism then M is injective. In [13, Theorem 1.6] it was shown that if M is injective and semi-finite then ρ is an isomorphism.

To complete this discussion let us show how an argument of S. Wassermann [32, proof of Theorem 6] makes it possible to remove the hypothesis that M is semi-finite. Wassermann's idea is to observe that since $N = M \rtimes_\sigma \mathbb{R}$, the crossed product of M by the modular automorphism group $\{\sigma_t\}$ of a faithful normal semi-finite weight, is semi-finite; one can use the result in the semi-finite case to prove the result in the type III case. Let us recall the notation.

We shall suppose that (H, J, P) is a standard form for M; and that M' denotes the commutant of M on this space. $\mathbb{C}(M, M')$ is the $*$-subalgebra of $\mathcal{L}(H)$ generated by M and M'. $\mathrm{C}^*(M, M')$ is the norm closure of $\mathbb{C}(M, M')$.

We define $\pi_\sigma : M \to \mathcal{L}(L^2(\mathbb{R}) \otimes H)$ by $[\pi_\sigma(x)\zeta](t) = \sigma_{-t}(x)\zeta(t)$ for $\zeta \in L^2(\mathbb{R}) \otimes H$, and $x \in M$. Let $[\lambda_t\zeta](s) = \zeta(s-t)$. Then $N = W^*(\pi_\sigma(M), \{\lambda_t\}_{t\in\mathbb{R}})$ is the crossed product $M \rtimes_\sigma \mathbb{R}$. For $x' \in M'$ let $\pi'_\sigma(x) = 1_{L^2(R)} \otimes x$. Then $\pi'_\sigma(M') \subseteq N'$. Moreover if $\tilde{J}\zeta(s) = \Delta_\varphi^{-is}(J(\zeta(s)))$ and $\tilde{P} = \{a\tilde{J}a\tilde{J}\xi \mid a \in N, \xi : \mathbb{R} \to P\}$ then $(\tilde{H}, \tilde{J}, \tilde{P})$ is a standard form for N and $\tilde{J}\pi_\sigma(x)\tilde{J} = \pi'_\sigma(JxJ)$.

Proposition 4.1

$M \otimes_{\mathrm{Z(M)}} M' \ni \Sigma x_i \otimes_{\mathrm{Z(M)}} y_i \mapsto \Sigma\pi_\sigma(x_i) \otimes_{\mathrm{Z(N)}} \pi'_\sigma(y_i) \in N \otimes_{\mathrm{Z(N)}} N'$ is an isometry when $M \otimes_{\mathrm{Z(M)}} M'$ has the $Z(M)$-min norm and $N \otimes_{\mathrm{Z(N)}} N'$ has the $Z(N)$-min norm.

Proof:

Observe that $\pi_\sigma(z) = \pi'_\sigma(z)$ for $z \in Z(M) = M \bigcap M'$. Hence the homomorphism, $M \otimes M' \ni \Sigma x_i \otimes y_i \mapsto \Sigma\pi_\sigma(x_i)\pi'_\sigma(y_i) \in \mathcal{L}(\tilde{H})$, is zero on the ideal $I_{Z(M)} \subseteq M \otimes M'$. Thus we have a map $M \otimes_{Z(M)} M' \ni \Sigma x_i \otimes_{Z(M)} y_i \longrightarrow \Sigma\pi_\sigma(x_i)\pi'_\sigma(y_i) \in \mathcal{L}(\tilde{H})$.

Note that for $\Sigma x_i \otimes y_i \in M \otimes M'$, $f \in L^2(\mathbb{R})$ and $\zeta \in H$ we have

$$(*) \qquad \|\Sigma\pi_\sigma(x_i)\pi'_\sigma(y_i)(f \otimes \zeta)\|^2 = \int_{-\infty}^{\infty} \|\Sigma\sigma_{-t}(x_i)y_i\zeta\|^2 |f(t)|^2 dt.$$

If $\Sigma x_i \otimes y_i \in M \otimes M'$ and $\Sigma\pi_\sigma(x_i)\pi'_\sigma(y_i) = 0$ then by $(*)$ $\|\Sigma\sigma_{-t}(x_i)\ y_i\zeta\| = 0$ $\forall\ \zeta, t$. Hence $\Sigma x_i y_i = 0$ (as an operator on H). Since $M \otimes_{\mathrm{Z(M)}} M' \simeq \mathbb{C}(M, M')$, we thus have that $\Sigma x_i \otimes_{Z(M)} y_i = 0$.

So $M \otimes_{\mathrm{Z(M)}} M' \ni \sum x_i \otimes_{Z(M)} y_i \mapsto \Sigma\pi_\sigma(x_i)\pi'_\sigma(y_i) \in \mathbb{C}(N, N')$ is a one-to-one homomorphism. Since $\mathbb{C}(N, N') \simeq N \otimes_{Z(M)} N'$; we thus have by composition $M \otimes_{Z(M)} M' \ni \Sigma x_i \otimes_{\mathrm{Z(M)}} y_i \mapsto \sum \pi_\sigma(x_i) \otimes_{Z(N)} \pi'_\sigma(y) \in N \otimes_{\mathrm{Z(N)}} N'$ is a one-to-one homomorphism.

By minimality of the norm $Z(M)$-min on $M \otimes_{Z(M)} M'$ we thus have that

$$(**) \qquad \|\sum x_i \otimes_{\mathrm{Z(M)}} y_i\|_{\mathrm{Z(M)\text{-}min}} \leq \|\sum \pi_\sigma(x_i) \otimes_{\mathrm{Z(N)}} \pi'_\sigma(y_i)\|_{\mathrm{Z(N)\text{-}min}}.$$

On $\mathcal{L}_{Z(N)}(\tilde{H}) \otimes \mathcal{L}_{Z(N)}(\tilde{H})$ we have two semi-norms $\|\cdot\|_{\mathrm{Z(N)\text{-}min}}$ and $\|\cdot\|_{\mathrm{Z(M)\text{-}min}}$. Since $Z(M) \subseteq Z(N)$ we have that for $\sum S_i \otimes T_i \in \mathcal{L}_{Z(N)}(\tilde{H}) \otimes \mathcal{L}_{Z(N)}(\tilde{H})$

$$\|\sum S_i \otimes T_i\|_{\mathrm{Z(N)\text{-}min}} \leq \|\sum S_i \otimes T_i\|_{\mathrm{Z(M)\text{-}min}}.$$

Hence $\|\sum \pi_\sigma(x_i) \otimes \pi'_\sigma(y_i)\|_{\mathrm{Z(N)\text{-}min}} \leq \|\sum \pi_\sigma(x_i) \otimes \pi'_\sigma(y_i)\|_{\mathrm{Z(M)\text{-}min}} = \|\sum x_i \otimes_{\mathrm{Z(M)}} y_i\|_{\mathrm{Z(M)\text{-}min}}$; the last equality coming from Theorem 3.3. Combining this with $(**)$ we have that

$$\|\sum x_i \otimes y_i\|_{\mathrm{Z(M)\text{-}min}} = \|\sum \pi_\sigma(x_i) \otimes \pi'_\sigma(y_i)\|_{\mathrm{Z(N)\text{-}min}}.$$

Proposition 4.2

If M is injective, then $M \otimes_{\mathrm{Z(M)\text{-}min}} M' \simeq \mathrm{C}^*(M, M')$.

Proof:

$N = M \times_\sigma \mathbb{R}$ is injective and semi-finite [30, Lemma 8.2]. So by [13, Theorem 1.6] $N \otimes_{\mathrm{Z(N)\text{-}min}} N' \simeq \mathrm{C}^*(N, N') \subseteq \mathcal{L}(\tilde{H})$. This combined with Proposition 4.1 shows that for $\sum x_i \otimes y_i \in M \otimes M'$ we have $\|\sum x_i \otimes y_i\|_{\mathrm{Z(M)\text{-}min}} = \|\Sigma \pi_\sigma(x_i) \otimes \pi'_\sigma(y_i)\|_{\mathrm{Z(N)\text{-}min}} = \|\sum \pi_\sigma(x_i)\pi'_\sigma(y_i)\|_{\tilde{H}}$.

So by $(*)$ (in the proof of Proposition 4.1) for $f \in L^2(\mathbb{R})$ and $\zeta \in H$ we have

$$\int_{-\infty}^{\infty} \|\sum \sigma_{-t}(x_i) y_i \zeta\|^2 |f(t)|^2 dt \leq \|f\|_2^2 \ \|\zeta\|^2 \ \|\sum x_i \otimes y_i\|^2_{\mathrm{Z(M)\text{-}min}}.$$

Thus for all $t \in \mathbb{R}$, $\|\Sigma \sigma_{-t}(x_i) y_i \zeta\| \leq \|\zeta\| \ \|\Sigma x_i \otimes y_i\|_{\mathrm{Z(M)\text{-}min}}$. So for $t = 0$ we have $\|\Sigma x_i y_i\| \leq \|\Sigma x_i \otimes y_i\|_{\mathrm{Z(M)\text{-}min}}$. Since the reverse inequality holds for any M (by minimality of $\|\cdot\|_{\mathrm{Z(M)\text{-}min}}$) we have the equality

$$\|\Sigma x_i y_i\|_H = \|\Sigma x_i \otimes y_i\|_{\mathrm{Z(M)\text{-}min}}.$$

Definition 4.3

Let M and N be von Neumann algebras and Z a common von Neumann subalgebra of their centres. Let π and σ be normal representations of M and N on a Hilbert space H, such that $\pi(M) \subseteq \sigma(N)'$ and $\pi(z) = \sigma(z)\ \forall z \in Z$. Then $\pi \otimes \sigma$ defines representation of $M \otimes_{\mathrm{Z}} N$ on H. We say that $\pi \otimes \sigma$ is *obtained from a commuting pair* *of representations which agree on Z.* For $x \in M \otimes_{\mathrm{Z}} N$, let $\|x\|_{\text{Z-bin}}$=sup$\{\|\pi(x)\| \mid \pi$ is obtained from a commuting pair of representations which agree on $Z\}$. Let $\overline{I_Z}^{\text{bin}}$ be the norm closure of I_Z in $M \otimes_{\text{Z-bin}} N$.

Lemma 4.4

For $x \in M \otimes N$, $\|x\|_{\text{Z-bin}} = \|x + \overline{I_Z}^{\text{bin}}\|_{\text{bin}}$.

Proof:

Any representation π, obtained from a commuting pair of normal representations which agree on Z defines a representation of $M \otimes_{\text{bin}} N$ which is zero on I_z; and thus π descends to a representation of the quotient, $M \otimes_{\text{bin}} N/\overline{I_Z}^{\text{bin}}$. Hence

$$\|\pi(x)\| \leq \|x + \overline{I_Z}^{\text{bin}}\|_{\text{bin}}$$

and so

$$\|x\|_{\text{Z-bin}} \leq \|x + \overline{I_Z}^{\text{bin}}\|_{\text{bin}}.$$

To prove the inequality in the other direction note that by Takesaki [31, Theorem III. 2.7] there is a central projection e, in $(M \otimes_{\text{bin}} N)^{**}$, such that $I_Z^{\perp} = e\,(M \otimes_{\text{bin}} N)^*$. Let $S_{\text{bin}}(M \otimes N)$ be the states on $M \otimes_{\text{bin}} N$ which are separately weak* continuous on $M \otimes N$. $S_{\text{bin}}(M \otimes N)$ is dense in $S(M \otimes_{\text{bin}} N)$. Hence $e\ S_{\text{bin}}(M \otimes N) \subseteq I_Z^{\perp}$. So if $\varphi \in S(M \otimes_{\text{bin}} N) \bigcap I_Z^{\perp}$ then φ is a limit of states $\{e\varphi_\alpha\} \subseteq S_{\text{bin}}(M \otimes N) \bigcap I_z^{\perp}$. Hence every state of $M \otimes_{\text{bin}} N/\overline{I_Z}^{\text{bin}}$ is bounded with respect to $\|\cdot\|_{\text{Z-bin}}$. Therefore

$$\|x + \overline{I_Z}^{\text{bin}}\|_{\text{bin}} \leq \|x\|_{\text{Z-bin}}.$$

Remark 4.5

Given φ a binormal state of $M \otimes N$ we have (following the notation of [11, §2]) $T_\varphi : M \to N_*$ defined by $T_\varphi(m)(n) = \varphi(m \otimes n)$. Note that $\varphi \in I_Z^{\perp}$ if and only if T_φ is Z-linear map of the Z-modules M and N_*.

Theorem 4.6

M is injective if and only if $\|\cdot\|_{\text{Z-bin}} = \|\cdot\|_{\text{Z-min}}$ on $M \otimes_{\mathrm{Z}} N$ for every unital von Neumann subalgebra Z of the centre of M and every von Neumann algebra N whose centre contains Z as a unital von Neumann subalgebra.

Proof:

Note that we always have the following commutative diagram

$$\begin{tikzcd} M \otimes_{\text{Z-bin}} M' \arrow[rr] \arrow[dr] & & M \otimes_{\text{Z-min}} M' \arrow[dl] \\ & \mathrm{C}^*(M, M') & \end{tikzcd}$$

Suppose that M has the property that for all N, $\|\cdot\|_{\text{Z-min}} = \|\cdot\|_{\text{Z-bin}}$ on $M \otimes N$. Letting $N = M'$ (with respect to a Hilbert space upon which M acts standardly) and $Z = Z(M)$; we have $\|\cdot\|_{\text{Z-bin}} = \|\cdot\|_{\text{Z-min}}$ on $M \otimes_{\mathrm{Z}} M'$. Hence $\mathrm{C}^*(M, M') \simeq M \otimes_{\text{Z-min}} M'$, and since $M \otimes_{\text{Z-min}} M'$ is a quotient of $M \otimes_{\min} M'$, M is injective by [11, Proposition 4.5].

Suppose M is injective, and let $\mathcal{E} : M \to Z$ be a normal conditional expectation of M onto Z. Let $F_{\mathcal{E}}$ be the set of maps $M \to M$ which can *approximately factored through* $\mathcal{E}$: $F_{\mathcal{E}} = \{\sum_{i,j} b_{ij}^* \mathcal{E}(a_{ij}^* \cdot a_{ij}) b_{ij} \mid a_{i,j}, b_{i,j} \in M\}$. Then id_M, the identity map $M \to M$, is a point-$\sigma(M, M_*)$-limit of completely positive maps in $F_{\mathcal{E}}$. If φ is a pure state on $M \otimes_{\text{Z-bin}} N$, $\Phi \in F_{\mathcal{E}}$, and $x \otimes y \in M \otimes N$ then $\varphi(\Phi(x) \otimes y) = \sum_{i,j} \varphi(b_{ij}^* \mathcal{E}(a_{ij}^* x a_{ij}) b_{ij} \otimes y) = \sum_{i,j} \mathcal{E}(a_{ij}^* x a_{ij})(\omega) \varphi(b_{ij}^* b_{ij} \otimes y)$, where $\varphi_{|Z}$ is evaluation at ω. Thus $\varphi \circ \Phi \otimes id_N$ is a sum of products of positive forms in $S(M_{\omega+}^*) \times S(N_{\omega+}^*)$ and hence is bounded in the norm $\|\cdot\|_{\text{Z-min}}$, since φ is a limit of states in $S(M \otimes_{\text{Z-min}} N)$, φ itself is in $S(M \otimes_{\text{Z-min}} N)$. Hence $\|\cdot\|_{\text{Z-bin}} = \|\cdot\|_{\text{Z-min}}$.

Theorem 4.7

Let M act standardly on H and $\mathbb{C}(M, M')$ be the *-algebra generated by M and M'. For ϑ, an automorphism of M which act trivially on $Z(M)$, let $\Theta : \mathbb{C}(M, M') \to \mathbb{C}(M, M')$ send $\sum_i m_i m_i'$ to $\sum_i \vartheta(m_i) m_i'$. If M is injective then Θ extends to an automorphism of $\mathrm{C}^*(M, M')$.

Proof: Since ϑ acts trivially on $Z(M)$, $\vartheta \otimes id_{M'}$ descends to an automorphism of $M \otimes_{\mathrm{Z}} M'$. Under the isomorphism of $M \otimes_{\mathrm{Z}} M'$ with $\mathbb{C}(M, M')$, $\vartheta \otimes id_{M'}$ is conjugate to Θ. By minimality of the norm $\|\cdot\|_{\text{Z-min}}$, $\vartheta \otimes id_{M'}$ extends to an automorphism of $M \otimes_{\text{Z-min}} M'$. By Proposition 4.2 $M \otimes_{\text{Z-min}} M'$ is isomorphic to $\mathrm{C}^*(M, M')$, and so Θ extends to an automorphism of $\mathrm{C}^*(M, M')$

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