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On the error term in a Parseval type formula in the theory of Ramanujan expansions II

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ABSTRACT

For two arithmetical functions f and g with absolutely convergent Ramanujan expansions, Murty and Saha have recently derived asymptotic formulas with error term for the convolution sum $\sum_{n\leq N} f(n)g(n+h)$ under some suitable conditions. In this follow up article we improve these results with a weakened hypothesis which is in some sense minimal.

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1. Introduction

In the seminal article [8], Ramanujan unfolded the theory of Ramanujan sums and Ramanujan expansions. He defined:

Definition 1. For positive integers r, n,

$$c_r(n) := \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^*} \zeta_r^{an},$$

where ζ_r denotes a primitive r-th root of unity.

Since then, these sums are attributed to Ramanujan and called Ramanujan sums. It is not hard to write $c_r(n)$ in terms of the Möbius function μ (see [6]). One has

$$c_r(n) = \sum_{d|n, d|r} \mu(r/d)d. \tag{1}$$

One also has the following explicit formula due to Hölder [4]:

$$c_r(n) = \frac{\varphi(r)}{\varphi(r/d)}\mu(r/d), \tag{2}$$

where $d = \gcd(n, r)$ and $\varphi(\cdot)$ denotes the Euler's totient function.

Ramanujan studied these sums in the context of point-wise convergent series expansion of the form $\sum_r a_r c_r(n)$ for various arithmetical functions. Such expansions are now known as Ramanujan expansions. More precisely:

Definition 2. We say an arithmetical function f admits a Ramanujan expansion if for each n, $f(n)$ can be written as a convergent series of the form

$$f(n) = \sum_{r \geq 1} \hat{f}(r) c_r(n)$$

for appropriate complex numbers $\hat{f}(r)$. The number $\hat{f}(r)$ is said to be the r-th Ramanujan coefficient of f with respect to this expansion.

Ramanujan himself observed [8] that such an expansion is not necessarily unique. He remarked that the assertion

$$\sum_{r \geq 1} \frac{c_r(n)}{r} = 0$$

is equivalent to the prime number theorem. This equation can be viewed as a Ramanujan expansion of the zero function.

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The vast archive on the theory of Ramanujan sums and Ramanujan expansions symbolize its great developments, in many directions with many different aspects, over the past 100 years or so. For instance, shortly after Ramanujan's death, Hardy [3] proved that

$$\frac{\phi(n)}{n}\Lambda(n) = \sum_{r=1}^{\infty} \frac{\mu(r)}{\phi(r)} c_r(n),$$

where Λ is the von Mangoldt function. The series on the right hand side is conditionally convergent and so is difficult to use. In [1], the authors showed that if we ignore convergence questions, Hardy's formula can be used to derive the Hardy–Littlewood conjecture about prime tuples. More precisely, one can derive the heuristic result that

$$\sum_{n\le N} \Lambda(n)\Lambda(n+h) \sim N \sum_{r=1}^{\infty} \frac{\mu^2(r)}{\phi(r)} c_r(h),$$

a conjecture formulated by Hardy and Littlewood using the more difficult circle method of Ramanujan. This led the authors of [2] to study convolution sums of the kind $\sum_{n\le N} f(n)g(n+h)$ for two arithmetical functions f and g with absolutely convergent Ramanujan expansions. They derived asymptotic formulas for such sums, which are analogous to Parseval's formula in the case of Fourier series expansions. However, the study of the error term for such formulas was not carried out there. It was then addressed in [7] by Murty and Saha. Under certain extended hypotheses they provide explicit error terms for such formulas.

The works [2] and [7] had some severe restrictions on the growth of the Ramanujan coefficients. The goal of this paper is to relax these conditions. To be precise, in [2] the authors had the following condition on the Ramanujan coefficients of f and g:

$$\sum_{r,s} \left|\hat{f}(r)\hat{g}(s)\right|(rs)^{1/2} d(r)d(s) < \infty,$$

which was then extended as

$$\left|\hat{f}(r)\right|, \left|\hat{g}(r)\right| \ll \frac{1}{r^{1+\delta}} \quad \text{for} \quad \delta > 1/2,$$

in [7]. The condition '$\delta > 1/2$', a priori seems to be ad hoc, as a similar condition for $\delta > 0$ would be sufficient to ensure the absolute convergence of the Ramanujan expansion. But if one wants to extend the hypothesis of [2], in the form that they have in [7], then '$\delta > 1/2$' is somewhat an optimal choice.

More recently, Saha [9] has considered the single sum $\sum_{n\le N} f(n)$ for an arithmetical function f with absolutely convergent Ramanujan expansion, in the context of deriving an asymptotic formula with explicit error term for such a sum. The author obtains his result under the above condition with $\delta > 0$. He also exhibits that even with the stronger

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condition that $\delta > 1/2$, one may end up getting a weaker result if the concerned sum is not handled carefully enough. This is exactly the phenomenon that we address here.

Hence, for our purpose we enforce the weakened (essentially a minimal) hypothesis '$\delta > 0$' (compared to both [2] and [7]), and still obtain a better error term. In the following section we state our results and compare them with their predecessors.

2. Statements of the theorems

In this article we prove the following theorems.

Theorem 1. *Suppose that f and g are two arithmetical functions with absolutely convergent Ramanujan expansions:*

$$f(n) = \sum_r \hat{f}(r) c_r(n), \qquad g(n) = \sum_s \hat{g}(s) c_s(n),$$

respectively. Further suppose that

$$\left|\hat{f}(r)\right|, \left|\hat{g}(r)\right| \ll \frac{1}{r^{1+\delta}}$$

for some $\delta > 0$. Then for a positive integer N, we have,

$$\sum_{n \le N} f(n)g(n) = \begin{cases} N \sum_{r \ge 1} \hat{f}(r)\hat{g}(r)\varphi(r) + O(N^{1-\delta}(\log N)^{4-2\delta}) & \text{if } \delta < 1, \\ N \sum_{r \ge 1} \hat{f}(r)\hat{g}(r)\varphi(r) + O(\log^3 N) & \text{if } \delta = 1, \\ N \sum_{r \ge 1} \hat{f}(r)\hat{g}(r)\varphi(r) + O(1) & \text{if } \delta > 1. \end{cases}$$

Remark 1. The exponent of N in the error term in the above theorem is consistent with the error term which has been obtained in [9] for sums of the form $\sum_{n \le N} f(n)$, which was not the case for the results in [7].

Theorem 2. *Let f and g be two arithmetical functions with the same hypotheses as in Theorem 1 and h be a positive integer. Then we have*

$$\sum_{n \le N} f(n)g(n+h) = \begin{cases} N \sum_{r \ge 1} \hat{f}(r)\hat{g}(r)c_r(h) + O(N^{1-\delta}(\log N)^{4-2\delta}) & \text{if } \delta < 1, \\ N \sum_{r \ge 1} \hat{f}(r)\hat{g}(r)c_r(h) + O(\log^3 N) & \text{if } \delta = 1, \\ N \sum_{r \ge 1} \hat{f}(r)\hat{g}(r)c_r(h) + O(1) & \text{if } \delta > 1. \end{cases}$$

Theorem 1 and Theorem 2 are substantially improved versions, in terms of both hypotheses and the conclusion, of the following two theorems respectively.

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Theorem 3. *(See Murty–Saha [7].) Let f and g be two arithmetical functions as in Theorem 1, with the last assumption being replaced by the condition that*

$$\left|\hat{f}(r)\right|, \left|\hat{g}(r)\right| \ll \frac{1}{r^{1+\delta}}$$

for some $\delta > 1/2$. Then one has,

$$\sum_{n\le N} f(n)g(n) = N\sum_{r} \hat{f}(r)\hat{g}(r)\varphi(r) + O\left(N^{\frac{2}{1+2\delta}}(\log N)^{\frac{5+2\delta}{1+2\delta}}\right).$$

Theorem 4. *(See Murty–Saha [7].) Let f and g be two arithmetical functions with the same hypotheses as in Theorem 3 and h be a positive integer. Then,*

$$\sum_{n\le N} f(n)g(n+h) = N\sum_{r} \hat{f}(r)\hat{g}(r)c_r(h) + O\left(N^{\frac{2}{1+2\delta}}(\log N)^{\frac{5+2\delta}{1+2\delta}}\right).$$

By virtue of Theorem 2, we can now naturally extend and improve the corollaries that were obtained in [7]. We quote:

Corollary 1. *For $s, t > 0$, let $\delta := \min\{s, t\}$. Then for any positive integer h we have*

$$\sum_{n\le N} \frac{\sigma_s(n)}{n^s}\frac{\sigma_t(n+h)}{(n+h)^t} = \begin{cases} N\frac{\zeta(s+1)\zeta(t+1)}{\zeta(s+t+2)}\sigma_{-(s+t+1)}(h) + O(N^{1-\delta}(\log N)^{4-2\delta}) & \text{if } \delta<1,\\ N\frac{\zeta(s+1)\zeta(t+1)}{\zeta(s+t+2)}\sigma_{-(s+t+1)}(h) + O(\log^3 N) & \text{if } \delta=1,\\ N\frac{\zeta(s+1)\zeta(t+1)}{\zeta(s+t+2)}\sigma_{-(s+t+1)}(h) + O(1) & \text{if } \delta>1, \end{cases}$$

where $\sigma_s(n) := \sum_{d|n} d^s$ and the big-oh constant depends on δ, hence on s and t.

This result has been stated in [5] only in the asymptotic form.

Corollary 2. *Let*

$$\phi_s(n) := n^s \prod_{p|n}(1-p^{-s}),$$

where the product is over prime divisors of n. Then for $s, t > 0$ and a positive integer h, we have

$$\sum_{n\le N} \frac{\phi_s(n)}{n^s}\frac{\phi_t(n+h)}{(n+h)^t} = \begin{cases} N\Delta(h) + O(N^{1-\delta}(\log N)^{4-2\delta}) & \text{if } \delta<1,\\ N\Delta(h) + O(\log^3 N) & \text{if } \delta=1,\\ N\Delta(h) + O(1) & \text{if } \delta>1, \end{cases}$$

where $\Delta(h) = \prod_{p|h}\left[\left(1-\frac{1}{p^{s+1}}\right)\left(1-\frac{1}{p^{t+1}}\right) + \frac{p-1}{p^{s+t+2}}\right]\prod_{p\nmid h}\left[\left(1-\frac{1}{p^{s+1}}\right)\left(1-\frac{1}{p^{s+1}}\right) - \frac{1}{p^{s+t+2}}\right]$, $\delta = \min\{s, t\}$ and the big-oh constant depends on δ, hence on s and t.

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3. Preliminaries

Some arguments for our proofs are provided by [7], which we discuss in the following section. In addition to that we need some well-known results which we record below.

Proposition 1. *For any real number $x \geq 1$,*

$$\sum_{k \leq x} \varphi(k) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Definition 3. The Mertens function $M(\cdot)$ is defined for all positive integers n as

$$M(n) := \sum_{k \leq n} \mu(k)$$

where $\mu(\cdot)$ is the Möbius function. The above definition can be extended to any real number $x \geq 1$ by defining

$$M(x) := \sum_{k \leq x} \mu(k).$$

Essentially from the error term in the prime number theorem one gets

Proposition 2. *For any real number $x \geq 1$,*

$$M(x) = \sum_{k \leq x} \mu(k) = O\left(x e^{-c\sqrt{\log x}}\right),$$

where c is some positive constant.

Let $d_k(n)$ be the number of ways of writing n as a product of k numbers. We generally write $d(n)$ to denote $d_2(n)$. Note that

$$d_4(n) = \sum_{\substack{a,b \\ ab=n}} d(a)d(b).$$

The functions $d_2(\cdot)$, $d_4(\cdot)$ appear in our proofs. Hence, we record the following general result about the average order of the arithmetical function $d_k(\cdot)$, which can be obtained by partial summation.

Proposition 3. *For any real number $x \geq 1$,*

$$\sum_{n \leq x} d_k(n) = \frac{x(\log x)^{k-1}}{(k-1)!} + O\left(x(\log x)^{k-2}\right).$$

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4. Proofs of the theorems

Fine-tuning arguments of [7] does not lead us to the desired results. However, to some extent, our proofs follow the arguments presented in [7] verbatim, but then major steps towards our desired results are taken by more detailed treatments for certain parts of the concerned sum. The proofs have two aspects of improvement. The improvement towards the error term is obtained by a finer analysis of a particular sum and then there is an elegant treatment of another sum which enables us to work with the weakened hypotheses. We elaborate below. To keep our exposition self-contained we recall relevant parts of the proof of Theorem 3 and Theorem 4 from [7].

4.1. Setting up the proofs

Here we explain the principle which is in the proof of Theorem 3 and Theorem 4, and also highlight the improvement towards the error term.

Let U be a parameter tending to infinity which is to be chosen later. We write,

$$\begin{aligned}\sum_{n\le N} f(n)g(n) &= \sum_{n\le N}\sum_{r,s} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n)\\ &= A+B, \quad \text{where}\end{aligned}$$

$$A := \sum_{n\le N}\sum_{\substack{r,s\\ rs\le U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n) \quad \text{and} \quad B := \sum_{n\le N}\sum_{\substack{r,s\\ rs>U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n).$$

To treat the sum A, we use the following lemma from [2].

Lemma 1. *Let h be a non-negative integer. Then*

$$\sum_{n\le N} c_r(n)c_s(n+h) = \delta_{r,s}Nc_r(h) + O(rs\log rs),$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta function.

Interchanging summations and applying Lemma 1 (for $h=0$), one gets upon separating $r=s$ and $r\ne s$,

$$\begin{aligned}A &= N\sum_{r^2\le U} \hat{f}(r)\hat{g}(r)\varphi(r) + O(U\log U)\\ &= C+D+O(U\log U), \quad \text{where}\end{aligned}$$

$$C = N\sum_{r} \hat{f}(r)\hat{g}(r)\varphi(r) \quad \text{and} \quad D = -N\sum_{r^2>U} \hat{f}(r)\hat{g}(r)\varphi(r).$$

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Clearly, C is the main term as per the theorem. Using the hypotheses and knowledge about the average order of the φ function (see Proposition 1), D is easily estimated (by partial summation) to be $O\left(\frac{N}{U^\delta}\right)$. Then one estimates the sum B to obtain the result. However, we do not go into the analysis of B which was done in [7], as a major improvement that we obtain here in this article is due to an independent treatment of the sum B.

When h is a positive integer, the sum $\sum_{n\le N} f(n)g(n+h)$ is also written as $A+B$, but here we have

$$A := \sum_{n\le N}\sum_{\substack{r,s\\ rs\le U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h) \quad\text{and}\quad B := \sum_{n\le N}\sum_{\substack{r,s\\ rs>U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h).$$

In this case, it turns out that, $A = C + D + O(U\log U)$, where $C = N\sum_r \hat{f}(r)\hat{g}(r)c_r(h)$, the main term and $D = -N\sum_{r^2>U}\hat{f}(r)\hat{g}(r)c_r(h)$. To estimate D one needs to know about $\sum_{r\le x} c_r(h)$, which is written as follows:

$$\sum_{r\le x} c_r(h) = \sum_{r\le x}\sum_{d|r,d|h}\mu(r/d)d = \sum_{\substack{k,d\\ dk\le x, d|h}} d\mu(k) = \sum_{d|h} d\sum_{k\le x/d}\mu(k).$$

The innermost sum is $M(x/d)$, where $M(\cdot)$ denotes the Mertens function. Using estimates on Mertens function (see Proposition 2), it is then obtained that $D = O\left(\frac{N\epsilon(h)}{U^{1/2+\delta}}\right)$, for some function $\epsilon(\cdot)$ of h which is bounded above by $e^{c\sqrt{\log h}}d(h)$ for some positive constant c.

However, one can do better with respect to the term $O(U\log U)$ above. Note that for the sum A, one actually has

$$A = C + D + O\left(\sum_{\substack{r,s\\ rs\le U}} \frac{1}{(rs)^{1+\delta}} rs\log rs\right).$$

The big-oh term can trivially be estimated to be $O(U\log U)$. We note that if $\delta > 1$, the sum $\sum_{r,s}\frac{1}{(rs)^\delta}\log rs$ is convergent. Hence the sum in that case is $O(1)$. For $0<\delta\le 1$, we can write

$$\sum_{\substack{r,s\\ rs\le U}} \frac{1}{(rs)^\delta}\log rs = \sum_{t\le U}\frac{d(t)\log t}{t^\delta} \le \log U\sum_{t\le U}\frac{d(t)}{t^\delta}.$$

Now from Proposition 3 (for $k=2$) we know that

$$\sum_{t\le U} d(t) = U\log U + O(U).$$

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Hence one can use partial summation to estimate the above sums. However, estimating $\sum_{t\le U}\frac{d(t)\log t}{t^\delta}$ is more complicated than estimating $\sum_{t\le U}\frac{d(t)}{t^\delta}$. Since we are only interested about the order of these sums, we will work with $\sum_{t\le U}\frac{d(t)}{t^\delta}$, as in both the cases the resulting order is the same.

Using partial summation one gets

$$\sum_{t\le U}\frac{d(t)}{t}=O(\log^2 U)\quad\text{and}\quad\sum_{t\le U}\frac{d(t)}{t^\delta}=O(U^{1-\delta}\log U),$$

for $\delta<1$. This clearly improves the exponent of U and yields,

$$A=\begin{cases}C+D+O(U^{1-\delta}\log^2 U) & \text{if } \delta<1,\\ C+D+O(\log^3 U) & \text{if } \delta=1,\\ C+D+O(1) & \text{if } \delta>1.\end{cases}$$

In the following two subsections we explain our approach of handling the sum B for $h=0$ and $h\neq 0$.

4.2. Proof of Theorem 1

Recall that by adapting the proof of Theorem 3 one can write $\sum_{n\le N}f(n)g(n)=A+B$, where

$$A:=\sum_{n\le N}\sum_{\substack{r,s\\ rs\le U}}\hat f(r)\hat g(s)c_r(n)c_s(n)\quad\text{and}\quad B:=\sum_{n\le N}\sum_{\substack{r,s\\ rs>U}}\hat f(r)\hat g(s)c_r(n)c_s(n).$$

In the previous subsection we obtained

$$A=\begin{cases}N\sum_{r\ge1}\hat f(r)\hat g(r)\varphi(r)+O\left(\frac{N}{U^\delta}\right)+O(U^{1-\delta}\log^2 U) & \text{if } \delta<1,\\ N\sum_{r\ge1}\hat f(r)\hat g(r)\varphi(r)+O\left(\frac{N}{U}\right)+O(\log^3 U) & \text{if } \delta=1,\\ N\sum_{r\ge1}\hat f(r)\hat g(r)\varphi(r)+O\left(\frac{N}{U^\delta}\right)+O(1) & \text{if } \delta>1.\end{cases}$$

Now here is the other major step towards improving the results of [7]. The essence of the proof lies in a careful analysis of the term B. Using (1) and the hypotheses about $\hat f(r)$, $\hat g(r)$, we write

$$\begin{aligned}B&=\sum_{n\le N}\sum_{\substack{r,s\\ rs>U}}\hat f(r)\hat g(s)c_r(n)c_s(n)\\ &\ll\sum_{\substack{r,s\\ rs>U}}\frac{1}{(rs)^{1+\delta}}\left|\sum_{r'|r}r'\mu(r/r')\sum_{s'|s}s'\mu(s/s')\sum_{\substack{m\le N/r'\\ r'm\equiv0 \bmod s'}}1\right|\end{aligned}$$

$$\leq \sum_{\substack{r,s \\ rs>U}} \frac{1}{(rs)^{1+\delta}} \sum_{r'|r} r' \sum_{s'|s} s' \sum_{\substack{m\leq N/r' \\ m\equiv 0 \bmod (s'/(r',s'))}} 1$$

$$\leq \sum_{\substack{r,s \\ rs>U}} \frac{1}{(rs)^{1+\delta}} \sum_{r'|r} r' \sum_{s'|s} s' \frac{N(r',s')}{r's'}.$$

Notice that

$$\sum_{r'|r} \sum_{s'|s} (r',s') \leq \sum_{l|r,l|s} l d(r/l) d(s/l).$$

This is because if $(r',s') = l$, then one can write $r' = lr''$ and $s' = ls''$. Now since $r'|r$ and $s'|s$, we get $r''|r/l$ and $s''|s/l$. Hence the number of choices of r'' and s'' are at most $d(r/l)$, $d(s/l)$ respectively. Thus we get, upon writing $r = lr_0$, $s = ls_0$ in the sum and interchanging summations,

$$B \ll N \sum_{l\geq 1} \frac{1}{l^{1+2\delta}} \sum_{\substack{r_0,s_0 \\ r_0s_0>U/l^2}} \frac{d(r_0)d(s_0)}{(r_0s_0)^{1+\delta}}.$$

Now we break the outermost sum into two parts, one for $l^2 > U$ and another for $l^2 \leq U$. If $l^2 > U$, the condition $r_0s_0 > U/l^2$ is vacuously true and then in that case the innermost sum is a convergent series for $\delta > 0$. So we deduce that

$$B \ll N \sum_{l\leq\sqrt{U}} \frac{1}{l^{1+2\delta}} \sum_{\substack{r_0,s_0 \\ r_0s_0>U/l^2}} \frac{d(r_0)d(s_0)}{(r_0s_0)^{1+\delta}} + N \sum_{l>\sqrt{U}} \frac{1}{l^{1+2\delta}}.$$

Note that the second sum is $O(\frac{N}{U^\delta})$ and also that the innermost sum in the first sum is nothing but

$$\sum_{t>U/l^2} \frac{d_4(t)}{t^{1+\delta}},$$

which by partial summation (and Proposition 3) turns out to be $O\left(\frac{\log^3(U/l^2)}{(U/l^2)^\delta}\right)$. Putting these informations together one gets

$$B \ll N \sum_{l\leq\sqrt{U}} \frac{1}{l^{1+2\delta}} \frac{\log^3(U/l^2)}{(U/l^2)^\delta} + \frac{N}{U^\delta} \ll \frac{N \log^4 U}{U^\delta}.$$

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Hence we obtain that

$$\sum_{n\le N} f(n)g(n) = \begin{cases} N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)\varphi(r) + O\left(\frac{N\log^4 U}{U^\delta}\right) + O(U^{1-\delta}\log^2 U) & \text{if } \delta<1,\\ N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)\varphi(r) + O\left(\frac{N\log^4 U}{U}\right) + O(\log^3 U) & \text{if } \delta=1,\\ N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)\varphi(r) + O\left(\frac{N\log^4 U}{U^\delta}\right) + O(1) & \text{if } \delta>1. \end{cases}$$

To optimize the error terms, we choose

$$U = \begin{cases} N\log^2 N & \text{if } \delta<1,\\ N\log N & \text{if } \delta=1,\\ N^{1/\delta}(\log N)^{4/\delta} & \text{if } \delta>1. \end{cases}$$

These choices yield,

$$\sum_{n\le N} f(n)g(n) = \begin{cases} N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)\varphi(r) + O(N^{1-\delta}(\log N)^{4-2\delta}) & \text{if } \delta<1,\\ N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)\varphi(r) + O(\log^3 N) & \text{if } \delta=1,\\ N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)\varphi(r) + O(1) & \text{if } \delta>1. \end{cases}$$

This concludes the proof of Theorem 1.

4.3. Proof of Theorem 2

We have already mentioned that we are only left to do a careful analysis of a particular sum, namely B. However in this case there are certain other difficulties, which we get around by further subdividing the sum B into parts concerning 'higher' and 'lower' values of s. One also needs Ingham's result [5] on the binary additive divisor problem. We elaborate below.

Keeping the earlier principle in mind, we write

$$\sum_{n\le N} f(n)g(n+h) = A + B, \quad \text{where}$$

$$A := \sum_{n\le N}\sum_{\substack{r,s\\ rs\le U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h) \quad \text{and} \quad B := \sum_{n\le N}\sum_{\substack{r,s\\ rs>U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h).$$

As per our derivations above, we have

$$A = \begin{cases} N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)c_r(h) + O\left(\frac{N\epsilon(h)}{U^{1/2+\delta}}\right) + O(U^{1-\delta}\log^2 U) & \text{if } \delta<1,\\ N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)c_r(h) + O\left(\frac{N\epsilon(h)}{U^{3/2}}\right) + O(\log^3 U) & \text{if } \delta=1,\\ N\sum_{r\ge 1}\hat{f}(r)\hat{g}(r)c_r(h) + O\left(\frac{N\epsilon(h)}{U^{1/2+\delta}}\right) + O(1) & \text{if } \delta>1. \end{cases}$$

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Now for the sum B, we have

$$B = \sum_{n\le N} \sum_{\substack{r,s\\ rs>U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h)$$

$$\ll \sum_{\substack{r,s\\ rs>U}} \frac{1}{(rs)^{1+\delta}} \left| \sum_{r'|r} r'\mu(r/r') \sum_{s'|s} s'\mu(s/s') \sum_{\substack{m\le N/r'\\ r'm\equiv -h \bmod s'}} 1 \right|.$$

Let $(r',s')=l$. Hence if we write $r'=lr''$ and $s'=ls''$ we get $(r'',s'')=1$, i.e., r'' is invertible mod s''. Also note that $l|h$. Thus

$$B \ll \sum_{\substack{r,s\\ rs>U}} \frac{1}{(rs)^{1+\delta}} \sum_{r'|r} r' \sum_{s'|s} s' \sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1$$

$$= \sum_{l|h} \sum_{\substack{r,s\\ rs>U\\ l|r,l|s}} \frac{1}{(rs)^{1+\delta}} \sum_{r''|r/l} lr'' \sum_{s''|s/l} ls'' \sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1.$$

Writing $r=lr_0$ and $s=ls_0$, we get

$$B \ll \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{\substack{r_0,s_0\\ r_0s_0>U/l^2}} \frac{1}{(r_0s_0)^{1+\delta}} \sum_{r''|r_0} r'' \sum_{s''|s_0} s'' \sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1$$

$$= E + F,$$

where

$$E := \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{\substack{r_0,s_0\\ r_0s_0>U/l^2}} \frac{1}{(r_0s_0)^{1+\delta}} \sum_{r''|r_0} r'' \sum_{\substack{s''|s_0\\ s''\le N/lr''}} s'' \sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1$$

and

$$F := \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{\substack{r_0,s_0\\ r_0s_0>U/l^2}} \frac{1}{(r_0s_0)^{1+\delta}} \sum_{r''|r_0} r'' \sum_{\substack{s''|s_0\\ s''>N/lr''}} s'' \sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1.$$

Note that

$$\sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1 \le \frac{N}{lr''s''} + 1.$$

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Hence if $s'' \leq N/lr''$, we get

$$\sum_{\substack{m \leq N/lr'' \\ r''m \equiv (-h/l) \bmod s''}} 1 \ll \frac{N}{lr''s''}.$$

This yields

$$E \ll N \sum_{l|h} \frac{1}{l^{1+2\delta}} \sum_{\substack{r_0, s_0 \\ r_0 s_0 > U/l^2}} \frac{d(r_0)d(s_0)}{(r_0 s_0)^{1+\delta}}.$$

Now the right hand side of the above expression is bounded by

$$N \sum_{l \geq 1} \frac{1}{l^{1+2\delta}} \sum_{\substack{r_0, s_0 \\ r_0 s_0 > U/l^2}} \frac{d(r_0)d(s_0)}{(r_0 s_0)^{1+\delta}},$$

which is $O\left(\frac{N \log^4 U}{U^\delta}\right)$ as we have seen in the proof of Theorem 1. Thus we finally have

$$E = O\left(\frac{N \log^4 U}{U^\delta}\right).$$

To treat the sum F, write $r_0 = r''n_r$ and $s_0 = s''n_s$. We then deduce

$$\begin{aligned} F &\leq \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{r_0, s_0} \frac{1}{(r_0 s_0)^{1+\delta}} \sum_{r''|r_0} r'' \sum_{\substack{s''|s_0 \\ s'' > N/lr''}} s'' \sum_{\substack{m \leq N/lr'' \\ r''m \equiv (-h/l) \bmod s''}} 1 \\ &\leq \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{r''} \frac{1}{(r'')^\delta} \left(\sum_{n_r} \frac{1}{n_r^{1+\delta}} \right) \sum_{s'' > N/lr''} \frac{1}{(s'')^\delta} \left(\sum_{n_s} \frac{1}{n_s^{1+\delta}} \right) \sum_{\substack{m \leq N/lr'' \\ r''m \equiv (-h/l) \bmod s''}} 1. \end{aligned}$$

The sums involving n_r and n_s are convergent, and so this simplifies to

$$\begin{aligned} F &\ll \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{r''} \frac{1}{(r'')^\delta} \sum_{s'' > N/lr''} \frac{1}{(s'')^\delta} \sum_{\substack{m \leq N/lr'' \\ r''m \equiv (-h/l) \bmod s''}} 1 \\ &\leq \sum_{l|h} \frac{1}{l^{2\delta}} \sum_{r''} \frac{1}{(r'')^\delta} \left(\frac{lr''}{N} \right)^\delta \sum_{s''} \sum_{\substack{m \leq N/lr'' \\ r''m \equiv (-h/l) \bmod s''}} 1 \\ &= \frac{1}{N^\delta} \sum_{l|h} \frac{1}{l^\delta} \sum_{r''} \sum_{s''} \sum_{\substack{m \leq N/lr'' \\ r''m \equiv (-h/l) \bmod s''}} 1. \end{aligned}$$

Now the sum

$$\sum_{r''}\sum_{s''}\sum_{\substack{m\le N/lr''\\ r''m\equiv(-h/l) \bmod s''}} 1=\sum_{n\le N/l} d(n)d(n+h/l).$$

From the solution to the binary additive divisor problem due to Ingham [5] we know that

$$\sum_{n\le N/l} d(n)d(n+h/l)\sim \frac{6}{\pi^2}\sigma_{-1}(h/l)\frac{N}{l}\log^2(N/l).$$

Thus, $F\ll N^{1-\delta}\log^2 N\sum_{l|h}\frac{\sigma_{-1}(h/l)}{l^{1+\delta}}$. Note that $\sigma_{-1}(n)=O(\log n)$. Hence

$$\sum_{l|h}\frac{\sigma_{-1}(h/l)}{l^{1+\delta}}=O(\log h).$$

Hence we obtain

$$F\ll_h N^{1-\delta}\log^2 N,$$

where the implied constant depends on h. Putting all these together and then for the same choice of U as in the proof of Theorem 1 we get

$$\sum_{n\le N} f(n)g(n+h)=\begin{cases} N\sum_{r\ge1}\hat{f}(r)\hat{g}(r)c_r(h)+O(N^{1-\delta}(\log N)^{4-2\delta}) & \text{if } \delta<1,\\ N\sum_{r\ge1}\hat{f}(r)\hat{g}(r)c_r(h)+O(\log^3 N) & \text{if } \delta=1,\\ N\sum_{r\ge1}\hat{f}(r)\hat{g}(r)c_r(h)+O(1) & \text{if } \delta>1.\end{cases}$$

This concludes the proof of Theorem 2.

5. Concluding remarks

Thus, our work (and in particular Corollary 1) gives an alternate and complete derivation (with an explicit error term) of one of Ingham's result [5], which was treated only for $s,t>1/2$ in [7]. It appears that the analysis of the error term in Ingham's work was never done before. The derivation of the main term in his theorem using the theory of Ramanujan sums was first initiated by the authors of [2]. An error term was then obtained in [7] as a further consequence of this theory, and for which we obtained a major improvement here.

The condition in our main theorems, namely

$$\left|\hat{f}(r)\right|,\left|\hat{g}(r)\right|\ll\frac{1}{r^{1+\delta}}$$

for $\delta > 0$ can be relaxed even further. This has recently been investigated in [10]. There is considerable importance in this because, as indicated in [2], Ramanujan expansions can be used to formulate the Hardy–Littlewood conjecture on twin primes.

Indeed, it may be possible to explore this further along the following lines. Let us recall the Ramanujan expansion for the von Mangoldt function $\Lambda(n)$ due to Hardy.

$$\frac{\varphi(n)}{n}\Lambda(n) = \sum_{r\geq 1} \frac{\mu(r)}{\varphi(r)} c_r(n).$$

For a positive real number s, we may define $\Lambda_s(n)$ by the following convergent sum.

$$\frac{\varphi(n)}{n}\Lambda_s(n) := \sum_{r\geq 1} \frac{\mu(r)}{\varphi(r) r^s} c_r(n).$$

Since

$$\frac{1}{\phi(r)} \ll \frac{\log\log r}{r},$$

the above sum converges absolutely for $s > 0$. Then our work can be applied to derive an asymptotic formula for

$$\sum_{n\leq N} \frac{\varphi(n)}{n}\Lambda_s(n)\frac{\varphi(n+h)}{n+h}\Lambda_s(n+h)$$

for any $s > 0$. The Hardy–Littlewood conjecture on prime 2-tuples is of course the case when $s = 0$ as mentioned in the introduction and we hope our work will converge in that direction.

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References

[1] H.G. Gadiyar, R. Padma, Ramanujan–Fourier series, the Wiener–Khintchine formula and the distribution of prime pairs, Phys. A 269 (1999) 503–510.
[2] H.G. Gadiyar, M. Ram Murty, R. Padma, Ramanujan–Fourier series and a theorem of Ingham, Indian J. Pure Appl. Math. 45 (5) (2014) 691–706.
[3] G.H. Hardy, Note on Ramanujan's trigonometrical function $c_q(n)$ and certain series of arithmetical functions, Proc. Cambridge Philos. Soc. 20 (1921) 263–271.
[4] O. Hölder, Zur Theorie der Kreisteilungsgleichung $K_m(x) = 0$, Pr. Mat.-Fiz. 43 (1936) 13–23.
[5] A.E. Ingham, Some asymptotic formulae in the theory of numbers, J. Lond. Math. Soc. 2 (3) (1927) 202–208.
[6] M. Ram Murty, Ramanujan series for arithmetical functions, Hardy-Ramanujan J. 36 (2013) 21–33.
[7] M. Ram Murty, B. Saha, On the error term in a Parseval type formula in the theory of Ramanujan expansions, J. Number Theory 156 (2015) 125–134.

[8] S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, Trans. Cambridge Philos. Soc. 22 (13) (1918) 259–276.
[9] B. Saha, On the partial sums of arithmetical functions with absolutely convergent Ramanujan expansions, Proc. Indian Acad. Sci. Math. Sci. (2015), in press.
[10] B. Saha, A note on arithmetical functions with absolutely convergent Ramanujan expansions, Int. J. Number Theory (2015), in press.

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