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The Generalized Dedekind Determinant

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Abstract. The aim of this note is to calculate the determinants of certain matrices which arise in three different settings, namely from characters on finite abelian groups, zeta functions on lattices and Fourier coefficients of normalized Hecke eigenforms. Seemingly disparate, these results arise from a common framework suggested by elementary linear algebra.

1. Introduction

The purpose of this note is three-fold. We prove three seemingly disparate results about matrices which arise in three different settings, namely from characters on finite abelian groups, zeta functions on lattices and Fourier coefficients of normalized Hecke eigenforms. In this section, we state these theorems. In Section 2, we state a lemma from elementary linear algebra, which lies at the heart of our three theorems. A detailed discussion and proofs of the theorems appear in Sections 3, 4 and 5.

In what follows below, for any $n \times n$ matrix A and for $1 \leq i, j \leq n$, $A_{i,j}$ or $(A)_{i,j}$ will denote the (i,j)-th entry of A. A diagonal matrix with diagonal entries $y_1, y_2, \ldots y_n$ will be denoted as $\operatorname{diag}(y_1, y_2, \ldots y_n)$.

Theorem 1.1. *Let $G = \{x_1, x_2, \ldots x_n\}$ be a finite abelian group and let $f : G \to \mathbb{C}$ be a complex-valued function on G. Let $\mathcal{F}$ be an $n \times n$ matrix defined by*

$$\mathcal{F}_{i,j} = f(x_i^{-1} x_j).$$

For a character χ on G, (that is, a homomorphism of G into the multiplicative group of the field $\mathbb{C}$ of complex numbers), we define

$$S_\chi := \sum_{s \in G} f(s)\chi(s).$$

The eigenvalues of $\mathcal{F}$ are S_χ as χ ranges over the irreducible characters of G. Thus, the determinant of $\mathcal{F}$ is equal to $\prod_\chi S_\chi$, where the product runs over all characters of G. Moreover, if $S_\chi \neq 0$ for every character χ of G, then $\mathcal{F}$ is invertible and

$$\mathcal{F}_{i,j}^{-1} = \frac{1}{n} \sum_{\chi} \frac{\chi(x_i x_j^{-1})}{S_\chi}.$$

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The next theorem indicates similarities between characters of finite abelian groups and Möbius functions on lattices (see Section 4 for detailed notation).

THEOREM 1.2. *Let $(L, \preccurlyeq)$ be a lattice on a set $\{1, 2, \ldots n\}$ of integers. Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be a complex valued function on the elements of L. We define $n \times n$ matrices C and D by*

$$C_{i,j} = f(i \wedge j),$$
$$D_{i,j} = f(i \vee j).$$

Then,

$$\det C = \prod_{i \leq n} \left(\sum_{j \preccurlyeq i} \mu(j,i) f(j) \right) \tag{1.1}$$

and

$$\det D = \prod_{i \leq n} \left(\sum_{i \preccurlyeq j} \mu(i,j) f(j) \right), \tag{1.2}$$

where $\mu(x,y)$ denotes the Möbius function on $(L, \preccurlyeq)$. Moreover, if $g(i) = \sum_{j \preccurlyeq i} \mu(j,i) f(j)$ and $h(i) = \sum_{i \preccurlyeq j} \mu(i,j) f(j)$ are non-zero for each $1 \leq i \leq n$, then C and D are invertible,

$$(C^{-1})_{i,j} = \sum_{i \vee j \preccurlyeq l} \frac{\mu(i,l)\mu(j,l)}{g(l)}. \tag{1.3}$$

and

$$(D^{-1})_{i,j} = \sum_{l \preccurlyeq i \wedge j} \frac{\mu(l,i)\mu(l,j)}{h(l)}. \tag{1.4}$$

Finally, the following theorem gives an interesting interpretation of recursive relations between Hecke operators acting on spaces of modular cusp forms.

THEOREM 1.3. *Let f be a normalized eigenform of weight k with respect to $\Gamma_0(N)$, for a positive integer N and a positive, even integer k. Let $f(z)$ have the Fourier expansion given by*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz),$$

where $e(z) = e^{2\pi i z}$ and $a_f(1) = 1$. Let A be an $n \times n$ matrix given by $A_{i,j} = a_f(ij)$. Then, the determinant of A is equal to $\mu(1)\mu(2)\ldots\mu(n)$, where $\mu(i)$ denotes the Möbius function on positive integers. Thus, A is non-invertible for any $n > 3$.

Special cases of Theorem 1.2 have been found by various authors since 1875, beginning with H.J.S. Smith [**S**] (see also the nice survey article [**H**]). The classical Smith determinant is

$$\det[(i,j)]_{n \times n} = \phi(1)\phi(2)\ldots\phi(n),$$

where (i,j) denotes the greatest common divisor of i and j. In 1977, Redheffer [**R**] discovered an interesting matrix related to the Riemann hypothesis. For each natural number n, define the matrix R_n such that $(R_n)_{i,j} = 1$ if $i|j$ or $j = 1$. Then,

$$\det(R_n) = M(n), \tag{1.5}$$

where

$$M(n) = \sum_{j \le n} \mu(j).$$

It is well-known that $M(n) = O(n^{1/2+\epsilon})$ for any $\epsilon > 0$ if and only if the Riemann hypothesis is true (see for example, [**T**]). Wilf [**W2**] generalized this result to any poset. In Section 6, we provide an alternate proof of Wilf's result, essentially following an elegant proof of Redheffer's theorem given in [**B**]. In Section 7, we also provide a link between Theorem 1.2 and chromatic polynomials.

2. A lemma from elementary linear algebra

In this section, we state a lemma, which helps us to develop a general setting from which Theorems 1.1, 1.2 and 1.3 arise as special cases. The advantage of this lemma is that it makes the calculation of inverses of the matrices in Theorems 1.1 and 1.2 amenable.

LEMMA 2.1. *Let A and B be square matrices of order n, such that the determinant of AB is equal to 1. Let $S = \{y_1, y_2, \dots y_n\}$ be a set of n complex numbers. Consider the matrix*

$$\mathfrak{D} = A \operatorname{diag}(y_1, y_2, \dots y_n) B.$$

Then, the determinant of $\mathfrak{D}$ is

$$\prod_{k=1}^{n} y_k.$$

Therefore, if $y_k \neq 0$ for every $1 \le k \le n$, then

$$\mathfrak{D}^{-1} = B^{-1} \operatorname{diag}\left(\frac{1}{y_1}, \frac{1}{y_2}, \dots \frac{1}{y_n}\right) A^{-1}.$$

In particular, if A is a unitary matrix, A^* denotes its conjugate transpose and $D = A \operatorname{diag}(y_1, y_2, \dots y_n) A^*$, with each $y_k \neq 0$, then

$$D^{-1} = A \operatorname{diag}\left(\frac{1}{y_1}, \frac{1}{y_2}, \cdots \frac{1}{y_n}\right) A^*.$$

3. The Dedekind determinant and proof of Theorem 1.1

Let $G = \{x_1, x_2, \dots x_n\}$ be a finite abelian group of order n. A character χ of G is a homomorphism of G into the multiplicative group of the field $\mathbb{C}$ of complex numbers. That is, $\chi : G \longrightarrow \mathbb{C}^*$ satisfies

$$\chi(ab) = \chi(a)\chi(b),\ a, b \in G.$$

It is well known that a finite abelian group of order n has exactly n distinct characters. Dedekind, in an unpublished work before 1896, made the following observation:

Let $f : G \longrightarrow \mathbb{C}$ be a complex valued function on G and let A be the $n \times n$ matrix whose (i, j)-th entry, as i and j vary between 1 and n, is given by

$$A_{i,j} = f(x_i^{-1} x_j).$$

Let

$$S_\chi = \sum_{s \in G} f(s)\chi(s).$$

The determinant of A is given by

$$\prod_{\chi} S_{\chi},$$

where the product runs over all characters on G. To see this, let v_{χ} denote the vector $\{\chi(x_1), \chi(x_2), \ldots \chi(x_n)\}$ for a given character χ of G. Observe that for each $1 \leq i \leq n$, the i-th element of Av_{χ} is given by

$$\begin{aligned}\sum_{j=1}^{n} f(x_i^{-1}x_j)\chi(x_j) &= \chi(x_i)\sum_{j=1}^{n} f(x_i^{-1}x_j)\chi(x_i^{-1}x_j) \\ &= \chi(x_i)S_{\chi},\end{aligned}$$

since $x_i^{-1}x_j$ runs over all elements of G as j varies from 1 to n. Thus, $Av_{\chi} = S_{\chi}v_{\chi}$. Since the distinct characters on G supply n linearly independent eigenvectors v_{χ} of A, the determinant of A is the product of the corresponding eigenvalues of A,

$$\prod_{\chi} S_{\chi}.$$

A is known as the Dedekind matrix and the determinant of A is called the Dedekind determinant. Thus A is invertible if and only if $S_{\chi} \neq 0$ for every character χ of G. The method of calculating the Dedekind determinant has many applications. For example, it can be used to determine the eigenvalues of the adjacency matrix of a Cayley graph, as shown in Section 2 of [**M2**]. It is also essential in the study of the regulator of cyclotomic extensions (see [**W**]).

In [**DGV**], the authors have applied this idea to calculate the determinant of the $n \times n$ matrix whose (i, j)-th element is given by

$$e\left(\frac{-si^{-1}j}{n}\right),$$

where $e(x) = e^{2\pi i x}$.

Given an integer $q \geq 2$, let $\mathbb{Z}_q$ denote the group of residue classes (mod q) and let $G(q)$ denote the multiplicative group of residue classes which are relatively coprime to q. Let

$$G(q) = \{x_1, x_2, \ldots x_{\phi(q)}\},$$

where $\phi(q)$ denotes the Euler-ϕ function. Given a function $F : \mathbb{Z}_q \longrightarrow \mathbb{C}$ with support in $G(q)$, the Fourier transform of F is defined by

$$\widehat{F}(n) = q^{-\frac{1}{2}} \sum_{m \in G(q)} F(m)e\left(\frac{-nm}{q}\right).$$

In order to determine whether it is possible to recover F from the values of $\widehat{F}$ restricted to $G(q)$, one has to check if the $\phi(q) \times \phi(q)$ matrix $\mathcal{F}_q$, whose (i, j)-th entry is

$$q^{-\frac{1}{2}}e\left(\frac{-x_i^{-1}x_j}{q}\right)$$

is invertible. $\mathcal{F}_q$ is a special case of the Dedekind matrix $\mathcal{F}$ where

$$f(x) = q^{-\frac{1}{2}}e\left(\frac{-x}{q}\right).$$

If $\mathcal{F}_q$ is invertible, the authors of [**DGV**] have explicitly constructed the inverse. However, by applying Lemma 2.1, we are able to generalise their result to an arbitrary complex-valued function defined on any finite abelian group.

Let χ_k, $1 \le k \le n$ denote the characters of a finite abelian group G of order n. For a complex-valued function f on G, let

$$S_k := \sum_{s \in G} f(s)\chi_k(s).$$

Observe that for any $g \in G$,

$$f(g) = \frac{1}{n}\sum_{k=1}^{n} S_k \chi_k(g^{-1}). \tag{3.1}$$

We define a matrix

$$A_{i,j} = \frac{\chi_j(x_i)}{\sqrt{n}},\ 1 \le i, j \le n.$$

Then, by equation (3.1),

$$\begin{aligned}(A\,\mathrm{diag}(S_1,\, S_2, \dots S_n)A^*)_{i,j} &= \frac{1}{n}\sum_{k=1}^{n}\chi_k(x_i)S_k\chi_k(x_j^{-1})\\ &= \frac{1}{n}\sum_{k=1}^{n} S_k\chi_k(x_i x_j^{-1})\\ &= f(x_i^{-1}x_j) = \mathcal{F}_{i,j}.\end{aligned}$$

We can now apply Lemma 2.1 to obtain Theorem 1.1.

In 1896, Dedekind wrote to Frobenius, suggesting the problem of calculating the determinant of matrices analogous to $\mathcal{F}$ for non-abelian groups. This is now recognized as the starting point of representation theory of finite groups, as remarked in [**CC**].

4. A combinatorial analog of the Dedekind determinant and proof of Theorem 1.2

In this section, we will develop formulae, similar to the ones in the previous section, in a combinatorial context.

Let $(L, \preccurlyeq)$ be a lattice on a set $\{1, 2, \cdots n\}$ of integers. For x, y in L, let $x \wedge y$ denote their greatest lower bound and let $x \vee y$ denote their least upper bound. An interval of L, $[x, y]$ is a set of the form $\{z \in L : x \preccurlyeq z \preccurlyeq y\}$. Let $I(L)$ be the set of intervals on L. We define the zeta function on $I(L)$ by

$$\zeta([x, y]) = \zeta(x, y) = \begin{cases} 1 & \text{if } x \preccurlyeq y, \\ 0 & \text{otherwise.}\end{cases}$$

We also define the Möbius function μ on $I(L)$ by the recursion

$$\sum_{x \preccurlyeq z \preccurlyeq y} \mu(x, z) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.}\end{cases}$$

It can be easily checked that

$$\sum_{x \preccurlyeq z \preccurlyeq y} \mu(x, z)\zeta(z, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.}\end{cases} \tag{4.1}$$

Now, let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be a complex valued function on the elements of L. We define $n \times n$ matrices C and D as follows :

For $1 \leq i, j \leq n$,

$$C_{i,j} = f(i \wedge j),$$
$$D_{i,j} = f(i \vee j).$$

In 1968, Wilf [**W1**] showed that

$$\det C = \prod_{i \leq n} \left(\sum_{j \preccurlyeq i} \mu(j,i) f(j) \right) \tag{4.2}$$

and

$$\det D = \prod_{i \leq n} \left(\sum_{i \preccurlyeq j} \mu(i,j) f(j) \right). \tag{4.3}$$

Notice that the factors in the above product play a role analogous to character sums S_χ in the Dedekind determinant. We will carry this analogy further and construct the inverse of C and D. Given a function f on L, we define a function $g : L \longrightarrow \mathbb{C}$ such that

$$f(x) = \sum_{y \preccurlyeq x} g(y).$$

Thus,

$$g(x) = \sum_{y \preccurlyeq x} \mu(y,x) f(y).$$

Also, define a function $h : L \longrightarrow \mathbb{C}$ such that

$$f(x) = \sum_{x \preccurlyeq y} h(y).$$

Then,

$$h(x) = \sum_{x \preccurlyeq y} \mu(x,y) f(y).$$

We recall that

$$i \vee j \preccurlyeq l \iff i \preccurlyeq l \text{ and } j \preccurlyeq l$$

and

$$t \preccurlyeq i \wedge k \iff t \preccurlyeq i \text{ and } t \preccurlyeq k.$$

We define a matrix Z by

$$Z_{i,j} = \zeta(j,i).$$

Equation (4.1) tells us that

$$Z^{-1}_{i,j} = \mu(j,i).$$

Since

$$f(i \wedge j) = \sum_{\substack{k \preccurlyeq i \\ k \preccurlyeq j}} g(k) = \sum_{k=1}^{n} \zeta(k,i) g(k) \zeta(k,j),$$

we have

$$C = Z \operatorname{diag}(g(1), g(2), \ldots g(n)) \, Z^T.$$

Thus, since Z is a unimodular matrix, by Lemma 2.1, the determinant of C is equal to $\prod_{k=1}^{n} g(k)$. Furthermore, if $g(k) \neq 0$ for every $1 \leq k \leq n$, C is invertible and

$$C^{-1} = (Z^T)^{-1} \operatorname{diag}\left(\frac{1}{g(1)}, \frac{1}{g(2)}, \dots \frac{1}{g(n)}\right) Z^{-1}.$$

That is,

$$C^{-1}_{i,j} = \sum_{k=1}^{n} \frac{\mu(i,k)\mu(j,k)}{g(k)}.$$

This proves equation (1.3). Similarly,

$$f(i \vee j) = \sum_{k=1}^{n} \zeta(i,k)h(k)\zeta(j,k).$$

Thus,

$$D = Z^T \operatorname{diag}\left(h(1), h(2), \dots h(n)\right) Z.$$

Just as above, this proves equation (1.4) and gives us Theorem 1.2.

We conclude this section with two applications of Theorem 1.2. The following special case of equation 1.3 appears as Problem 32 in Chapter 8 of [**PS**].

Let L be a lattice on $\{1, 2, \dots n\}$ with the natural ordering, that is,

$$i \preccurlyeq j \text{ if } i \leq j.$$

With this ordering, $i \wedge j$ is the minimum of i and j, and $i \vee j$ is the maximum of i and j. Let $a_1, a_2, \dots a_n$ be a set of n complex numbers. For each $1 \leq i \leq n$, we define

$$f(i) = a_1 + a_2 + \cdots + a_i.$$

Clearly, for each i lying between 1 and n,

$$\sum_{1 \leq j \leq i} \mu(j,i) f(j) = a_i.$$

Thus, with C defined as before, applying equation (1)(proved by Wilf in [**W1**]), we get that

$$\det C = a_1 a_2 \dots a_n.$$

In particular, in the special case $a_i = 1$ for all i, we see that the matrix whose (i,j)-th entry is $\min(i,j)$, has determinant 1.

Moreover, if each a_i is non-zero, then C is invertible and by Theorem 1.2, for every $1 \leq i, j \leq n$, the (i,j)-th entry of C^{-1} is given by

$$\sum_{i,j \leq l \leq n} \frac{\mu(i,l)\mu(j,l)}{a_l}.$$

There are not many summands in this expression since $\mu(i,l) = 0$ unless $l = i$ or $l = i+1$. Indeed, it is easy to see that the sum is zero if $|i-j| \geq 2$. Otherwise, the sum equals

$$\begin{cases} \dfrac{1}{a_n} & \text{if } i = j = n, \\ -\dfrac{1}{a_{i+1}} & \text{if } i - j = -1,\ 1 \leq i \leq n-1, \\ \dfrac{1}{a_i} + \dfrac{1}{a_{i+1}} & \text{if } 1 \leq i = j \leq n-1, \\ -\dfrac{1}{a_i} & \text{if } i - j = 1,\ 1 \leq i \leq n. \end{cases}$$

That is, the inverse matrix is a tri-diagonal matrix.

If we consider the cognate matrix D whose (i, j)-th entry is $\max(i, j)$, then we need to calculate the function h given by

$$h(x) = \sum_{x \leq y} \mu(x, y) f(y).$$

In our special situation $f(y) = y$ and so $h(x) = f(x) - f(x+1) = -1$ if $x < n$ and n if $x = n$. This leads to the determinant of D being $(-1)^{n-1}n$.

Now, we will look at another special case of equation (1), which appears as Problem 33 in Chapter 8 of [**PS**].

If L is the lattice on $\{1, 2, \dots n\}$ ordered by divisibility, then

$$i \wedge j = (i, j),$$

the greatest common divisor of i and j. Also,

$$i \vee j = [i, j],$$

the least common multiple of i and j. Define $f(n) = n$ for all $n \in \mathbb{N}$. Then $g(n) = \phi(n)$ since

$$\sum_{d|n} \phi(d) = n.$$

From Theorem 1.2, we deduce that the determinant of C in this case is

$$\phi(1)\phi(2) \dots \phi(n)$$

and $C_{i,j}^{-1}$ is given by

$$\sum_{l \leq n,\, [i,j]|l} \frac{1}{\phi(l)} \mu\left(\frac{l}{i}\right) \mu\left(\frac{l}{j}\right).$$

The Smith determinant suggests we look at the determinant of the matrix T whose (i, j)-th entry is given by the least common multiple of i, j, denoted $[i, j]$. This can be deduced from our formalism above. However, there is a simpler way to evaluate it. Let $g(d)$ be such that

$$\sum_{d|n} g(d) = \frac{1}{n}.$$

By Möbius inversion, we have

$$g(n) = \sum_{d|n} \mu(d) d/n = \frac{1}{n} \prod_{p|n} (1 - p).$$

Thus,

$$\frac{1}{(i, j)} = \sum_{d|(i,j)} g(d)$$

so that

$$[i, j] = \frac{ij}{(i, j)} = \sum_{d|i, d|j} i g(d) j,$$

which we can view from the perspective of Lemma 4 and deduce

$$T = A \operatorname{diag}(g(1), \dots, g(n)) B,$$

where A is the matrix whose d,i-th entry is i if $d|i$ and zero otherwise, and B is simply the transpose of A. As A has determinant $n!$, and so does B, we see that

$$\det T = n!^2 g(1)\cdots g(n).$$

This does not fit directly into the format of our Lemma 4 as A and B do not satisfy $\det(AB) = 1$, but it is clear that the arrangement does.

5. A modular analogue of the Dedekind determinant

Let f be a normalized eigenform of weight k with respect to $\Gamma_0(N)$. Let $f(z)$ have the Fourier expansion given by

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz),$$

where $e(z) = e^{2\pi i z}$ and $a_f(1) = 1$. The Fourier coefficients of such an eigenform satisfy a recursive relation given by

$$a_f(m)a_f(n) = \sum_{d|(m,n)} a_f\left(\frac{mn}{d^2}\right). \tag{5.1}$$

By Möbius inversion (see Lemma 2.1 of [**M1**]), one gets

$$a_f(mn) = \sum_{d|(m,n)} \mu(d) a_f\left(\frac{m}{d}\right) a_f\left(\frac{n}{d}\right).$$

We now define, for $n \geq 1$, a matrix U given by

$$U = \begin{cases} a_f\left(\frac{j}{i}\right), & \text{if } i|j, \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the matrix given by

$$A_{i,j} = a_f(ij),\ 1 \leq i,j \leq n.$$

Then, by equation (5.1),

$$A = U \operatorname{diag}(\mu(1),\, \mu(2), \ldots \mu(n))\, U^T,$$

where U is a matrix with determinant 1. Once again, as an immediate application of Lemma 2.1, we derive Theorem 1.3. This tells us that A has determinant 0 for any $n > 3$ and for a normalised Hecke eigenform of any weight and level.

6. The Redheffer matrix revisited

Following [**B**], we prove Redheffer's theorem and adapt our proof to prove Wilf's theorem which generalized Redheffer's result to the context of partially ordered sets. We prove the following:

THEOREM 6.1. *Let R_n be the $n \times n$ matrix whose (i,j)-th entry is* 1 *if $i|j$ or if $j = 1$. Then,*

$$\det R_n = \sum_{j \leq n} \mu(j).$$

Proof. Let S be the $n \times n$ matrix whose (i,j)-th entry is 1 if $i|j$ and zero otherwise. Let T be the $n \times n$ matrix whose (i,j)-th entry is $M(n/i)$ if $j = 1$, 1 if $i = j \geq 2$ and 0 otherwise. We claim that $R_n = ST$. Indeed, the (i,j)-th entry of the product is

$$\sum_{k=1}^{n} S_{ik} T_{kj} = \sum_{i|k} T_{kj}.$$

For $j = 1$, this sum is

$$\sum_{i|k} M(n/k) = \sum_{t \leq n/i} M(n/it) = 1,$$

by an elementary result in number theory. Indeed, we have

$$\sum_{d|n} \mu(d) = 0$$

unless $n = 1$ in which case it is equal to 1. Thus,

$$1 = \sum_{j \leq n} \sum_{de=j} \mu(d) = \sum_{e \leq n} \sum_{d \leq n/e} \mu(d) = \sum_{e \leq n} M(n/e).$$

For $j \geq 2$, the sum is 1 if $i|j$ and zero otherwise. This completes the proof. □

This argument generalizes to posets and one can construct an analog of the Redheffer matrix as follows:

Theorem 6.2. *Let $\{x_1, x_2, \ldots x_n\}$ be a finite partially ordered set with order denoted by $\preccurlyeq$ such that x_1 is the minimal element of this poset. Let $\mathbf{R_n}$ be the $n \times n$ matrix whose (i,j)-th entry is $\zeta(x_i, x_j)$ if $i \leq j$ or if $j = 1$. If $i > j$, the (i,j)-th entry of $\mathbf{R_n}$ is 0. Then,*

$$\det \mathbf{R_n} = \sum_{\substack{m \leq n \\ x_1 \preccurlyeq x_m}} \mu(x_1, x_m).$$

Remark 6.3. *The above theorem was proved by Wilf in* [**W1**]. *However, we provide a simpler proof by generalising ideas from the proof of Theorem* 6.1 *to posets.*

Proof. Let $\mathbf{S}$ be the $n \times n$ matrix whose (i,j)-th entry is $\zeta(x_i, x_j)$ if $i \leq j$ and 0 if $i > j$. Let $\mathbf{T}$ be the $n \times n$ matrix whose (i,j)-th entry is

$$\sum_{\substack{m \leq n \\ x_i \preccurlyeq x_m}} \mu(x_i, x_m)$$

if $j = 1$, 1 if $i = j \geq 2$ and 0 otherwise.

If $j = 1$, the (i,j)-th entry of the product $\mathbf{ST}$ is seen to be

$$\sum_{\substack{k \leq n \\ x_i \preccurlyeq x_k}} \sum_{\substack{m \leq n \\ x_k \preccurlyeq x_m}} \mu(x_k, x_m).$$

On interchange of summation, this equals

$$\sum_{\substack{m \leq n \\ x_i \preccurlyeq x_m}} \left(\sum_{\substack{k \leq n \\ x_i \preccurlyeq x_k \preccurlyeq x_m}} \mu(x_k, x_m) \right).$$

Since the inner sum equals 1 if $i = m$ and 0 otherwise, we see that the $(i, 1)$-th entry of the product $\mathbf{ST}$ equals the $(i, 1)$-th entry of $\mathbf{R_n}$. It is trivial to check that the other entries of the matrices $\mathbf{R_n}$ and $\mathbf{ST}$ match. We have

$$\det \mathbf{R_n} = \det \mathbf{T} = \sum_{\substack{m \le n \\ x_1 \preccurlyeq x_m}} \mu(x_1, x_m),$$

since the determinant of $\mathbf{S}$ is 1. □

7. Link with chromatic polynomials

If M is a planar map with $r(M)$ regions, we can colour this map using λ colours in $\lambda^{r(M)}$ ways. Among these colourings, only a subset are proper colourings. Any colouring can be reduced to a proper colouring of a unique submap (simply by "erasing" the common boundary between two regions receiving the same colouring). Thus, if $P_M(\lambda)$ is the number of proper colourings of M using λ-colours, we get

$$\lambda^{r(M)} = \sum_{B \subseteq M} P_B(\lambda),$$

where B runs over all submaps of M. If we think of the collection of such submaps B's as a partially ordered set with respect to set inclusion, Möbius inversion gives us

$$P_M(\lambda) = \sum_{B \subseteq M} \mu(B, M) \lambda^{r(B)}.$$

If we apply Theorem 1.2 to calculate the determinant of the matrix C with $f(x) = \lambda^{r(x)}$ associated to the poset of submaps of a planar map M, we find that the determinant vanishes if there is no proper colouring of M using λ colours.

8. Concluding remarks

There have been several papers addressing the problem of determining the eigenvalues of the Redheffer matrix, notably due to Vaughan ([**V1**], [**V2**]) and others. Vaughan [**V1**] computed the characteristic polynomial of R_n and showed that R_n has exactly $n - [\log_2 n] - 1$ of its eigenvalues equal to unity. R_n also has two "dominant" eigenvalues which are approximately equal to $\pm\sqrt{n}$. The size of the remaining $[\log_2 n] - 1$ eigenvalues, known as the non-trivial eigenvalues, has been further investigated by Vaughan [**V2**] as well as Barrett and Jarvis [**BJ**]. Our approach through linear algebra to the determinant of R_n and related determinants shows that there is an underlying linear algebra theme to all of them and perhaps, viewing classical open questions such as the Riemann hypothesis from this perspective will shed more light on it. Also, viewing the four color problem in this context may open up a more conceptual approach to its solution.

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