Applications of Symmetric Power L-functions

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Preface

These lectures were given at the Fields Institute as part of a one-semester program on automorphic forms from January 2003 to July 2003. They were addressed to graduate students and non-specialists who wanted to see how the exotic theory of the Langlands program can be applied to classical questions arising in analytic number theory. As such, we have never aimed for completeness or for technical precision. Rather, we aimed to convey how the methods can be applied to such questions. Wherever possible, we have given precise references for the interested reader so that he/she may fill in the required details, if needed.

It is assumed that the reader has some familiarity with analytic and algebraic number theory, though often, we give a quick review. For the most part, the lectures can be read easily and some understanding can be gained of how the role analytic number theory plays in the Langlands program.

We use as motivating themes, the Sato-Tate conjecture, the Ramanujan conjecture, the Selberg eigenvalue conjecture and the celebrated Artin's conjecture about holomorphy of non-abelian L-series. There are further applications but time has not permitted us to go into details. In some places, other applications are briefly indicated.

These lectures complement the other two courses of Cogdell and Kim. Indeed, the technical details of the Rankin-Selberg method as well as converse theory are recurring themes of these lectures also. Moreover, the holomorphy of symmetric power L-functions combined with sophisticated methods of averaging from analytic number theory imply surprising results that often have important consequences to questions with a classical flavour. Wherever possible, we indicate some of these consequences and indicate how further refinements can be made.

It should be stressed that these lectures are somewhat informal and in some places lack "textbook rigor". Nevertheless, we hope they will be useful to both student and researcher alike.

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LECTURE 1

The Sato-Tate Conjecture

1 Introduction

There are many significant applications of the theory of symmetric power L-functions to questions arising from classical analytic number theory. In these notes, we will touch upon only a few of them. In this lecture, we will discuss the Sato-Tate conjecture and discuss the relationship between this conjecture and the analyticity of the symmetric power L-functions. In the next lecture, we will discuss the Ramanujan conjecture and the Selberg eigenvalue conjecture.

Let E be an elliptic curve over a number field F. For each prime ideal v of F where E has good reduction, the number of points of E mod v is given by

$$N(v) + 1 - a_v$$

where $N(v)$ denotes the norm of v and a_v satisfies Hasse's inequality

$$|a_v| \leq 2(N(v))^{1/2}.$$

Thus, we can write

$$a_v = 2N(v)^{1/2} \cos \theta_v$$

for a uniquely defined angle θ_v satisfying $0 \leq \theta_v < \pi$. The Sato-Tate conjecture is a statement about how the angles θ_v are distributed in the interval $[0, \pi]$ as v varies.

To study the distribution of the angles θ_v attached to an elliptic curve, we have to consider two cases. The first case is when the elliptic curve has CM (complex multiplication). This refers to the well-known fact that the ring of endomorphisms of an elliptic curve E is either isomorphic to the ring of ordinary integers or is an order in an imaginary quadratic field k. In the former case we say E has no CM (no complex multplication) and in the latter case, we say E has CM.

Let us now look at the CM case. For simplicity, let us suppose that k is contained in F, the field over which E is defined. Then, the sequence $\{\theta_v, -\theta_v\}$, as v ranges over the places of F, is uniformly distributed in $[-\pi, \pi]$. If F does not contain k, the situation is a little more complicated with a slightly different density function that has been determined (see [39]).

In the non-CM case, the distribution is unknown at present. We will show below that the angles are **not** uniformly distributed when $F = \mathbb{Q}$. Sato and Tate (independently) predicted another law of distribution for the angles θ_v. More precisely, they predict that

$$\#\{v : N(v) \leq x : \theta_v \in (\alpha, \beta)\} \sim \left(\frac{2}{\pi} \int_\alpha^\beta \sin^2 \theta d\theta \right) \pi_F(x)$$

as x tends to infinity, where $\pi_F(x)$ is the number of prime ideals of F whose norm is less than x.

2 Uniform distribution

We will begin with a general discussion of the classical setting for uniform distribution. A sequence of real numbers $\{x_n\}$ is called **uniformly distributed** (modulo 1) if for any pair of real numbers α, β with $0 \leq \alpha < \beta < 1$, we have

$$\#\{n \leq N : \quad x_n \in (\alpha, \beta)\} \sim (\beta - \alpha)N$$

as N tends to infinity.

Theorem 1.1 (Weyl's Criterion) *The sequence $\{x_n\}$ is uniformly distributed mod 1 if and only if for all $m \geq 1$,*

$$\sum_{n \leq N} e^{2\pi i m x_n} = o(N)$$

as N tends to infinity.

Proof (Sketch) First suppose that the sequence is uniformly distributed. We will show the condition is necessary. Let us observe that any continuous function f can be approximated by a linear combination of step functions so that for any given $\epsilon > 0$, we have

$$\sup_{x \in [0,1]} |f(x) - \sum_i c_i \chi_{I_i}(x)| \leq \epsilon,$$

where χ_I denotes the characteristic function of the interval I. Then,

$$\sum_{n \leq N} f(x_n) = \sum_i c_i \left(\sum_{n \leq N} \chi_{I_i}(x_n) \right) + O(\epsilon N).$$

By hypothesis,

$$\sum_{n \leq N} \chi_{I_i}(x_n) = \mu(I_i)N + o(N),$$

where $\mu(I)$ denotes the measure of the interval I. Now the sum

$$\sum_i c_i \mu(I_i)$$

is a Riemann sum and as our epsilon gets smaller, the sum converges to the integral

$$\int_0^1 f(x)dx.$$

Thus, we have proved that

$$\frac{1}{N} \sum_{n \leq N} f(x_n) \to \int_0^1 f(x)dx.$$

In particular, we can apply this to $\cos mx$ and $\sin mx$ to deduce the required result.

For the converse, we approximate $\chi_I(x)$ by trigonometric polynomials (which can be done by the Stone-Weierstrass theorem). In fact, one can be more precise. For any positive integer K, there are trigonometric polynomials $m(x)$ and $M(x)$ of degree $\leq K$ such that

$$m(x) \leq \chi_I(x) \leq M(x)$$

with

$$m(x) = \sum_{|m| \leq K} a_m e^{2\pi i m x}, \quad M(x) = \sum_{|m| \leq K} b_m e^{2\pi i m x}$$

with

$$a_0 = b_0 = \mu(I) + O(1/K).$$

Therefore,

$$\#\{n \le N : \quad x_n \in I\} = \sum_{n \le N} \chi_I(x_n) = \mu(I)N + o(N),$$

as required. □

Theorem 1.1 says that to establish uniform distribution of the angles θ_v, we need to study the exponential sums

$$\sum_{N(v) \le x} e^{2\pi i m \theta_v}.$$

In the CM case, Hecke proved a theorem that implies that the series

$$L(s,\chi) := \prod_v \left(1 - \frac{\chi(v)}{N(v)^s}\right)^{-1}$$

with $\chi(v) = e^{2\pi i \theta_v}$, extends to an entire function for $\Re(s) \ge 1$ and does not vanish there. The same applies to $L(s,\chi^m)$ for each natural number m. Thus, we can now apply a classical Tauberian argument to deduce the uniform distribution of the θ_v. We briefly review the relevant theorem in the next section.

3 Wiener-Ikehara Tauberian theorem

Theorem 1.2 *Let $f(s) = \sum_{n=1}^\infty a_n/n^s$, with $a_n \ge 0$, and $g(s) = \sum_{n=1}^\infty b_n/n^s$ be two Dirichlet series with $|b_n| \le a_n$ for all n. Assume that $f(s)$ and $g(s)$ extend analytically to $\Re(s) \ge 1$ except possibly at $s = 1$ where they have a simple pole with residues R and r (which may be zero) respectively. Then*

$$\sum_{n \le x} b_n \sim rx$$

as x tends to infinity.

The classical application of this theorem is the deduction of the prime number theorem. Let

$$f(s) = -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n) = \log p$ when $n = p^a$ for some prime p and zero otherwise. Taking $g(s) = f(s)$ in the above theorem allows us to deduce the prime number theorem

$$\sum_{n \le x} \Lambda(n) \sim x$$

using the well-known fact that the Riemann zeta function does not vanish on $\Re(s) = 1$.

One can apply this theorem to $L(s,\chi^m)$ above and deduce the uniform distribution of the angles after a routine application of partial summation.

In a fundamental paper written in 1970, Langlands [30] outlined an approach to the Sato-Tate conjecture using the theory of automorphic forms. (It is possible that some of these ideas may have had roots in the earlier work of Sato and Tate.) To simplify matters and notation, we will give only a rough outline of this approach and relegate to later lectures the more precise details.

Firstly, Langlands suggested the automorphic viewpoint. Thus, the conjecture of Sato-Tate was applicable in a larger context of modular forms, or more generally, to automorphic forms on $GL(2)$. For example, one could take the celebrated Ramanujan τ function attached to the unique newform of weight 12 and level 1, and write

$$\tau(p) = 2p^{11/2}\cos\theta_p.$$

One expects the same Sato-Tate distribution for these angles θ_p as well.

Here is a brief description of the strategy of Langlands [30]. For each natural number m, put

$$L_m(s) = \prod_v \prod_{j=0}^{m} \left(1 - \frac{\alpha_v^{m-j}\beta_v^j}{Nv^s}\right)^{-1}$$

where $\alpha_v = e^{i\theta_v}, \beta_v = e^{-i\theta_v}$. Langlands indicated that the theory of automorphic forms predicts that each $L_m(s)$ should extend to an entire function. In fact, if each $L_m(s)$ extends analytically for $\Re(s) \geq 1$, and does not vanish there, then by the Tauberian theorem, the Sato-Tate conjecture follows. Kumar Murty [39] showed that the non-vanishing hypothesis can be dispensed with because a very elegant argument extending the classical one of Hadamard and de la Vallée Poussin allows one to show non-vanishing from having analytic continuation to $\Re(s) \geq 1$.

In the case F is the rational number field, it is now a theorem due to Wiles and others that $L_1(s)$ is essentially the L-function attached by Hecke to a classical cusp form of weight 2. Thus, in this particular case, the Langlands conjecture is established. The non-vanishing of $L_1(s)$ on $\Re(s) = 1$, is a result due to Rankin. For $m = 2$, Rankin-Selberg theory allows one to deduce that $L_2(s)$ extends to an entire function for $\Re(s) \geq 1$. The continuation of $L_2(s)$ to the entire complex plane was established by Shimura [59] in the case $F = \mathbb{Q}$ and in the general case by Gelbart and Jacquet [10]. In very recent work, Kim and Shahidi [23] showed that $L_3(s)$ extends to an entire function and later, Kim, showed the same for $L_4(s)$. For the cases $5 \leq m \leq 9$, Kim and Shahidi have shown that $L_m(s)$ extends to a meromorphic function for all $s \in \mathbb{C}$ which is regular for $\Re(s) \geq 1$, except in the case of $m = 9$, $L_9(s)$ may have a pole at $s = 1$.

Let us remark that Rankin's result on $L_2(s)$ is already sufficient to show that in the non-CM case, the Sato-Tate distribution does not hold. Also, if $L_9(s)$ were to have a pole at $s = 1$, then the Sato-Tate conjecture would be false, as we will indicate below.

4 Weyl's theorem for compact groups

Serre [53] gave the following reformulation of the Weyl criterion for uniform distribution in the context of a compact group. Let G be a compact group and X its space of conjugacy classes. Let μ denote its normalised Haar measure. A sequence of elements $\{x_n\}$ with $x_n \in X$ is said to be uniformly distributed in X if for every continuous function f with compact support, we have

$$\sum_{n \leq N} f(x_n) \sim N \int_X f d\mu$$

as N tends to infinity.

Theorem 1.3 (Weyl's criterion for compact groups) *Let G be a compact group with Haar measure μ. A sequence $\{x_n\}$ is uniformly distributed in G if and only if*

$$\sum_{n\leq N}\chi(x_n)=o(N)$$

for every irreducible character χ of G.

The classical case in Theorem 1.1 corresponds to $G=\mathbb{R}/\mathbb{Z}$ because in this case, the irreducible characters are given by $x\mapsto e^{2\pi imx}$.

Serre gave an interesting reformulation of this criterion in the context of L-functions. Let F be a field and for each place v of F, let $x_v\in G$. For each irreducible representation $\rho:G\to GL_n(\mathbb{C})$, we let

$$L(s,\rho)=\prod_v \det(I-\rho(x_v)Nv^{-s})^{-1}.$$

Theorem 1.4 (Serre) *Suppose that for each irreducible non-trivial representation ρ of G, the L-function $L(s,\rho)$ extends to an analytic function for $\Re(s)\geq 1$. Then, the sequence $\{x_v\}$ is uniformly distributed in X if and only if $L(s,\rho)$ does not vanish on $\Re(s)=1$.*

In the context of the Sato-Tate conjecture, one considers the group $SU(2,\mathbb{C})$ where the conjugacy classes are parametrized by

$$X_\theta=\begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix},\quad 0\leq\theta\leq\pi.$$

The image of the Haar measure in the space of conjugacy classes of $SU(2,\mathbb{C})$ is known to be

$$\frac{2}{\pi}\sin^2\theta d\theta.$$

The irreducible representations of $SU(2,\mathbb{C})$ are the symmetric power representations ρ_m of the standard representation ρ_1 of $SU(2,\mathbb{C})$ into $GL(2,\mathbb{C})$. We find that $L(s,\rho_m)$ as defined above by Serre coincide with $L_m(s)$ defined in section 3.

Since tr $\rho_m(X_\theta)=\sin(m+1)\theta/\sin\theta$, the Sato-Tate conjecture is equivalent to the assertion

$$\sum_{N(v)\leq x}\frac{\sin(m+1)\theta_v}{\sin\theta_v}=o(\pi_F(x)),$$

for each natural number m. So far, this has been established only for $m\leq 8$ by the work of Kim and Shahidi [23].

LECTURE 2

Maass Wave Forms

1 Maass forms of weight zero

If we consider modular forms without the holomorphy condition but insist that our function is an eigenfunction of the non-Euclidean Laplacian:

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

we arrive at the notion of real analytic forms. We may write such a function, as a function of the variables x, y and since $f(z+1) = f(z)$, we have

$$f(x, y) = \sum_n a_n(f, y) e^{2\pi i n x}.$$

Suppose that $\Delta f = \lambda f$. This gives us a condition on the coefficients $a_n(f, y)$, namely that they satisfy

$$-y^2 \frac{d^2}{dy^2} a_n(f, y) = (\lambda - 4\pi^2 n^2 y^2) a_n(f, y).$$

One can renormalize and show that

$$f(x, y) = a_0(f) y^s + a_0'(f) y^{1-s} + \sum_{n \neq 0} a_n(f) \sqrt{y} K_{ir}(2\pi |n| y) e^{2\pi i n x}$$

where

$$K_{ir} = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y \cosh t - irt} dt$$

with $\lambda = 1/4 + r^2$.

Maass proved that the series

$$\sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}$$

extends to a meromorphic function for all $s \in \mathbb{C}$ analytic everywhere except possibly at $s = 0$ and $s = 1$, and satisfies a functional equation.

We have the celebrated Ramanujan conjecture that for any $\epsilon > 0$, $a_n(f) = O(n^\epsilon)$. The Selberg conjecture is that $\lambda \geq 1/4$, or equivalently, r is real and not purely imaginary.

In his 1970 paper, Langlands [30] interprets the Selberg conjecture as a Ramanujan conjecture "at infinity" and thus puts both conjectures on an equal conceptual footing. This viewpoint has roots in earlier work of Satake.

By the work of Kim and Shahidi [24], we know that $a_n = O(n^{7/64})$ and that $\lambda \geq .238$ by the recent work of Kim and Sarnak [22]

2 Maass forms with weight

Let us fix a discrete subgroup Γ of $SL_2(\mathbb{R})$. Here we consider functions on the extended upper half plane which satisfy the following:

(i) $f(\gamma z) = ((c\overline{z}+d)/|cz+d|)^k f(z)$ for all

$$\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma;$$

(ii) f is an eigenfunction of

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\frac{\partial}{\partial x}$$

(iii) a growth condition of the form $f(x+iy) = O(y^N)$ for some $N > 0$ as y tends to infinity.

One can show the existence of "shift" operators that will reduce the study of these spaces essentially to the study of weight zero or weight one Maass forms. Thus, often in the literature, (see for example [4]) the focus is on weight zero or the weight one case.

If f is a classical modular form of weight k, then it is not hard to show that $y^{k/2}f(z)$ is a Maass form of weight k with eigenvalue $k(2-k)/4$. Therefore, the study of Maass forms includes the study of modular forms from this perspective.

The set of Maass forms of fixed weight and eigenvalue is a vector space over $\mathbb{C}$. Moreover, we have an involution acting on this space given by the map

$$\iota : f(z) \to f(-\overline{z}).$$

A form is called even if $\iota \circ f = f$ and odd if $\iota \circ f = -f$. Therefore, the space of Maass forms decomposes as a direct sum of two subspaces consisting of even forms and odd forms respectively.

The L-series

$$\sum_{n=1}^{\infty}\frac{a_n(f)}{n^s}$$

extends to an entire function and satisfies a functional equation:

$$\begin{aligned} &Q^s\Gamma\left(\frac{s+\delta+r}{2}\right)\Gamma\left(\frac{s+\delta-r}{2}\right)L(s,f) \\ = \;& wQ^{1-s}\Gamma\left(\frac{1-s+\delta-r}{2}\right)\Gamma\left(\frac{1-s+\delta+r}{2}\right)L(1-s,\overline{f}) \end{aligned}$$

where $\delta = 0$ or 1 according as f is even or odd and $Q > 0$.

For $\Gamma = SL_2(\mathbb{Z})$, Selberg [52] proved that $\lambda \geq 1/4$ and this was extended to congruence subgroups of sufficiently small level by Vignéras [60]. For general arithmetic groups, Selberg showed that $\lambda \geq 3/16$.

3 Eisenstein series

The simplest example of a Maass form is given by the Eisenstein series

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{(m,n)\neq(0,0)}\frac{y^s}{|mz+n|^{2s}}.$$

This series converges for $\Re(s) > 1$ and we clearly have

$$E(\gamma z, s) = E(z,s)$$

for all $\gamma \in SL_2(\mathbb{Z})$. In addition, it is easily verified that

$$\Delta E(z,s) = s(1-s)E(z,s)$$

so that $E(z,s)$ is a weight zero Maass form with eigenvalue $s(1-s)$. Since $E(z,s)$ is periodic with period 1, we can derive its Fourier series:

$$E(z,s) = \sum_{r=-\infty}^{\infty} a_r(y,s)e^{2\pi irx}$$

and

$$a_r(y,s) = \int_0^1 E(x+iy,s)e^{-2\pi irx}dx.$$

We do the obvious. We insert the series expansion for $E(z,s)$ into the integral and apply Fubini's theorem. First, the contribution to $E(z,s)$ from $m=0$ is

$$\pi^{-s}\Gamma(s)y^s\zeta(2s).$$

This is part of $a_0(y,s)$ but not all of a_0 as we shall see below. Now suppose $m \neq 0$. Since (m,n) and $(-m,-n)$ give the same summand in $E(z,s)$, we may suppose $m > 0$. Thus,

$$a_r(y,s) = \pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int_0^1 [(mx+n)^2+m^2y^2]^{-s}e^{-2\pi irx}dx.$$

If we put $n = qm+d$ with $0 \le d < m$, the sum becomes

$$\sum_{m=1}^{\infty}\sum_{d \bmod m}\int_{-\infty}^{\infty}[(mx+d)^2+m^2y^2]^{-s}e^{-2\pi irx}dx.$$

We change variables: $x = u - d/m$ to get

$$\sum_{m=1}^{\infty} m^{-2s}\int_{-\infty}^{\infty}(u^2+y^2)^{-s}e^{-2\pi iru}\left(\sum_{d \bmod m} e^{2\pi idr/m}\right)du.$$

The innermost sum is zero unless $m|r$ in which case it is m. Thus, the sum becomes

$$\sum_{m|r} m^{1-2s}\int_{-\infty}^{\infty}(u^2+y^2)^{-s}e^{-2\pi iru}du$$

If $r = 0$, we get

$$\pi^{-s}\Gamma(s)y^s\zeta(2s-1)\int_{-\infty}^{\infty}(u^2+y^2)^{-s}du$$

which is equal to

$$\pi^{-s}\sqrt{\pi}\Gamma(s-1/2)y^{1-s}\zeta(2s-1).$$

Thus, the constant term (on applying the functional equation for $\zeta(s)$) is equal to

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}.$$

If $r \neq 0$, then we get

$$a_r(y,s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y)$$

where

$$\sigma_{1-2s}(r) = \sum_{m|r} m^{1-2s}.$$

One can show that $a_r(y,s) = a_r(y,1-s)$ and $r^s\sigma_{-2s}(r) = r^{-s}\sigma_{2s}(r)$ from which the functional equation is easily deduced.

4 Upper bound for Fourier coefficients and eigenvalue estimates

We begin with the elementary observation

$$e^{-1/x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)x^s ds$$

which is easily demonstrated by contour integration and Stirling's formula. Hence,

$$\sum_{n=1}^{\infty} a_n e^{-n/x} = \frac{1}{2\pi i}\int_{(2)} \Gamma(s)f(s)x^s ds$$

where

$$f(s) = \sum_{n=1}^{\infty} a_n/n^s.$$

Now suppose that $a_n \geq 0$ and $f(s)$ is absolutely convergent for $\Re(s) \geq 1+\epsilon$. Moving the line of integration to $\Re(s) = 1+\epsilon$ gives

$$\sum_{n=1}^{\infty} a_n e^{-n/x} = O(x^{1+\epsilon}).$$

Thus, for any individual term in the sum, we have

$$a_n e^{-n/x} = O(x^{1+\epsilon}).$$

Choosing $x = n$, we deduce that $a_n = O(n^{1+\epsilon})$.

It may look as if we were wasteful in the above analysis and a finer argument would give a better estimate. This, however, is not true as can be seen by considering

$$f(s) = \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^{ks}} = \zeta(ks-k+1).$$

In this example, we have

$$a_n = \Omega(n^{1-\epsilon})$$

for any $\epsilon > 0$ and so, we cannot reduce the exponent in the penultimate analysis.

Now consider

$$L_m(s) := L(s,\pi,r_m) = \prod_p \prod_{j=1}^{m}\left(1 - \frac{\alpha_p^{m-j}\beta_p^j}{p^s}\right)^{-1}$$

where we are ignoring the finitely many Euler factors that need to be modified corresponding to the ramified factors.

Consider the L-function

$$L(s,\pi,r_m \otimes \overline{r}_m) = \prod_{k\leq 2m,\, k \text{ odd}} L(s,\pi,r_k).$$

The proof of this identity is equivalent to the trigonometric identity

$$1 + \frac{\sin 3\theta}{\sin\theta} + \frac{\sin 5\theta}{\sin\theta} + \cdots \frac{\sin(2m-1)\theta}{\sin\theta} = \left(\frac{\sin m\theta}{\sin\theta}\right)^2$$

which is easily proved by induction and left as an exercise for the reader.

Thus, the series $L(s, \pi, r_m \otimes \overline{r}_m)$ is a Dirichlet series with non-negative coefficients. If we now suppose that for each $m \geq 1$, $L(s, \pi, r_m)$ is analytic for $\Re(s) \geq 1 + \epsilon$, then its p-th coefficient (for p prime) is $O(p^{1+\epsilon})$ by the argument given above. But the p-th coefficient is easily calculated to be

$$\left| \sum_{j=1}^{m} \alpha_p^{m-j} \beta_p^j \right|^2 .$$

Moreover, $|\alpha_p \beta_p| = 1$ so that if the Ramanujan conjecture is false, one of these has absolute value greater than 1. Without any loss of generality, suppose it is α_p. Then, in the above summation, α_p^m dominates the sum so we deduce

$$|\alpha_p|^{2m} = O(p^{1+\epsilon}).$$

Taking, $2m$-th roots, we obtain

$$\alpha_p = O(p^{(1+\epsilon)/2m}),$$

and letting m tend to infinity, we obtain $\alpha_p = O(1)$ which is the Ramanujan conjecture.

As we remarked above, this reasoning cannot be sharpened. However, using the fact that each of the L-functions $L(s, \pi, r_m)$ satisfies a functional equation, one can improve the estimate using a classical result of Chandrasekharan and Narasimhan [5]. This result says that if $a_n \geq 0$ and $f(s) = \sum_{n=1}^{\infty} a_n / n^s$ is convergent in some half-plane, has analytic continuation for all s except for a pole at $s = 1$ of order k and it satisfies a functional equation of the form

$$Q^s \Delta(s) f(s) = w Q^{1-s} \Delta(1-s) f(1-s)$$

where $Q > 0$ and

$$\Delta(s) = \prod_i \Gamma(\alpha_i s + \beta_i)$$

then

$$\sum_{n \leq x} a_n = x P_{k-1}(\log x) + O(x^{\frac{2A-1}{2A+1}} \log^{k-1} x)$$

where $A = \sum_i \alpha_i$. Taking differences, we deduce that

$$a_n = O(n^{\frac{2A-1}{2A+1}} \log^{k-1} n).$$

In [42], this result is stated with a typo on page 525. (On lines 3 and 7 of [42], $(2A-1)(2A+1)$ should be $(2A-1)/(2A+1)$ in both instances.)

A similar reasoning can be applied to obtain bounds in the Selberg eigenvalue conjecture. If π corresponds to a Maass form with eigenvalue λ, then the Gamma factors in the functional equation of $L(s, \pi, r_m)$ will have the following shape:

$$\Gamma(s, \pi, r_m) = \prod_{j=0}^{m} \Gamma\left(\frac{s - \lambda_j}{2}\right), \quad \lambda_j = i(m-2j)r, \quad \lambda = \frac{1}{4} + r^2.$$

One can also study oscillations of Fourier coefficients of modular forms as well as Dirichlet series constructed out of Kloosterman sums. This we will take up in later lectures.

LECTURE 3

The Rankin-Selberg Method

1 Eisenstein series and non-vanishing of $\zeta(s)$ on $\Re(s) = 1$

I want to indicate a proof of the non-vanishing of $\zeta(s)$ on $\Re(s) = 1$ which uses the theory of Eisenstein series and as a consequence does not use the Euler product of $\zeta(s)$ as most conventional proofs do. The idea was used by Jacquet and Shalika [17] in their general result about the non-vanishing on $\Re(s) = 1$ of automorphic L-functions associated with GL_n.

Recall that

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}.$$

Notice that we may also write this as

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s$$

where Γ_∞ is the stabilizer of the cusp at infinity.

We showed last time that

$$\begin{aligned} E(z,s) &= \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s} \\ &\quad + \sum_{r\neq 0} |r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y)e^{2\pi i r x} \end{aligned}$$

where $\sigma_v(n) = \sum_{d|n} d^v$ and

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\frac{dt}{t}.$$

One can prove directly that $K_s(y) = K_{-s}(y)$ and $r^s\sigma_{-2s}(r) = r^s\sigma_{2s}(r)$ which allows us to deduce the functional equation of $E(z,s)$ from its Fourier expansion.

This result lies at the heart of the Langlands-Shahidi method of analytic continuation of Eisenstein series. It is also at the core of the Rankin-Selberg method of analytic continuation which we outline below.

Now suppose that $\zeta(1+it_0) = 0$ for some t_0 real. Then, $\zeta(1-it_0) = 0$ also. We put $s = (1+it_0)/2$ in $E(z,s)$. Then, the constant term vanishes and we get a Maass cusp form:

$$E(z,(1+it_0)/2) = 4\sqrt{y}\sum_{r=1}^\infty r^{it_0/2}\sigma_{-it_0}(r)\cos(2\pi r x)\int_0^\infty e^{-\pi r y(t+t^{-1})}\frac{dt}{t^{1-it_0/2}}.$$

Using standard estimates for the integral, one can show that the sum is $O(e^{-cy})$ for some $c > 0$. Hence the constant term of $E(z,(1+it_0)/2)$ is zero and we have a genuine Maass cusp form on our hands.

In particular,

$$\int_0^1 E(x+iy,(1+it_0)/2)dx = 0.$$

Mulitplying this equation by y^{s-2} and integrating from 0 to ∞, we get

$$\int_0^\infty \int_0^1 E(x+iy,(1+it_0)/2)y^{s-2}dxdy = 0.$$

Now we use the fundamental idea that

$$\cup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma(\Gamma\backslash H) = [0,1]\times[0,\infty],$$

usually referred to as the "unfolding" of the domain of integration. Thus,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma(\Gamma\backslash H)} E(z,(1+it_0)/2)\Im(z)^s\frac{dxdy}{y^2} = 0.$$

As $E(\gamma z,s) = E(z,s)$, we may change variables and get:

$$0 = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash H} E(z,(1+it_0)/2)\Im(\gamma z)^s\frac{dxdy}{y^2} = \int_{\Gamma\backslash H} E(z,(1+it_0)/2)E(z,s)\frac{dxdy}{y^2},$$

valid for all $s\in\mathbb{C}$.

From the definition of $E(z,s)$ (or its Fourier expansion) we see that

$$E(z,\bar{s}) = \overline{E(z,s)}.$$

Therefore, putting $s=(1-it_0)/2$, we get from the penultimate equation,

$$0 = \int_{\Gamma\backslash H} |E(z,(1+it_0)/2)|^2\frac{dxdy}{y^2}.$$

Thus, the integrand is identically zero. That is, we have proved that $\zeta(1+it_0)=0$ implies that

$$E(z,(1+it_0)/2)\equiv 0.$$

We now show that this is a contradiction. We do this by showing that some Fourier coefficient of $E(z,(1+it_0)/2)$ is non-zero. That is, we need to check

$$\int_0^\infty e^{-\pi ry(u+u^{-1})}\frac{du}{u^{1+it_0}} \neq 0.$$

If we set $u=e^\theta$, we have to show that

$$\int_{-\infty}^\infty e^{-\pi ry(e^\theta+e^{-\theta})-it_0\theta}d\theta \neq 0.$$

In other words, it suffices to show that

$$\int_0^\infty e^{-\pi ry(e^\theta+e^{-\theta})}\cos t_0\theta d\theta \neq 0.$$

This integral is of the form

$$\int_0^\infty e^{-y(a^\theta+a^{-\theta})}\cos\theta d\theta, \quad a>1.$$

We would like to determine its behaviour as y tends to infinity. To do this, we can apply Laplace's saddle point method: if f has two continuous derivatives, with $f(0) = f'(0) = 0$ and $f''(0) > 0$, and f is increasing in $[0, A]$, then

$$I(x) := \int_0^A e^{-xf(t)} dt \sim \sqrt{\frac{\pi}{2xf''(0)}}$$

as x tends to infinity and provided $I(x_0)$ exists for some x_0. A slightly generalized version of this says that if g is continuous on $[0, A]$, then

$$\int_0^A g(t)e^{-xf(t)} dt \sim g(0)\sqrt{\frac{\pi}{2xf''(0)}}.$$

Now choose $f(t) = a^t + a^{-t} - 2$, $g(t) = \cos t$ so that

$$e^{-2x} \int_0^\infty e^{-x(a^\theta + a^{-\theta} - 2)} \cos\theta d\theta \sim e^{-2x} 2\log a \sqrt{\frac{\pi}{x}}$$

from which we see that $E(z, (1 + it_0)/2) \not\equiv 0$, as required. This gives the desired contradiction.

It is possible to deduce the non-vanishing of the above integrals directly without appealing to Laplace's saddle point method. With some work, it may also be possible to derive a zero-free region for $\zeta(s)$. These ideas are further developed in the PhD thesis of M. McKee.

2 Explicit construction of Maass cusp forms

The first examples of Maass cusp forms were constructed by Maass [34] in 1949. Alternate treatments of this subject can also be found in [4] and [37].

Let F be a quadratic field over $\mathbb{Q}$ with narrow class number one. (This means that the order of the narrow ideal class group is one, where the equivalence relation for narrow ideals is modulo principal ideals with a totally positive generator.) Let ψ be a Hecke character. Such a character has the form $\psi = \psi_\infty \psi_f$ for some some finite order character ψ_f with conductor f. We will consider only characters with $f = \mathbf{O}_F$ so that $\psi(\mathfrak{a}) = \psi_\infty(\alpha)$ where α is a totally positive generator of $\mathfrak{a}$. Let v and e be as follows: v is purely imaginary, and e, equals 0 or 1. Then

$$\psi_\infty(x) = \mathrm{sgn}(x_1)^e \mathrm{sgn}(x_2)^e |x_1/x_2|^v$$

where x_1 and x_2 are the Galois conjugates of x. It is necessary to have that $\psi_\infty(\eta) = 1$ for $\eta \in \mathbf{O}_F^\times$. The fact that F has narrow class number one implies there is a fundamental unit $\epsilon > 1$ whose norm is -1. This forces $v = mi\pi/2\log\epsilon$ with m an ordinary integer. If $m \neq 0$, we get a family of Maass cusp forms:

$$\theta_\psi(z) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})\sqrt{y} K_v(2\pi N(\mathfrak{a})y) \cos 2\pi N(\mathfrak{a})x$$

if $e = 0$.

If $e = 1$, we may take

$$\theta_\psi(z) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})\sqrt{y} K_v(2\pi N(\mathfrak{a})y) \sin 2\pi N(\mathfrak{a})x.$$

Maass [34] (see p. 112 of [4] also) shows that each of these is a cusp form for $\Gamma_0(D)$ where D is the discriminant of the quadratic field F. The corresponding eigenvalue is

$$\frac{1}{4} + \frac{m^2\pi^2}{4(\log\epsilon)^2}.$$

This construction is really a special case of Langlands functoriality, namely automorphic induction.

The fact that θ_ψ is a Maass form is proved using converse theory in pp. 112-118 of [4]. In general, one expects a map

$$\mathbf{A}(K) \to \mathbf{A}(k)$$

from the space of automorphic representations of $GL_n(\mathbb{A}_K)$ to the space of automorphic representations of $GL_{nd}(\mathbb{A}_k)$ where $d = [K : k]$ where the map is given as follows. Let Π be a cuspidal automorphic representation of K and suppose

$$L(s, \Pi) = \prod_w L(s, \Pi_w),$$

where the product is over all places w of K. One expects that there is a π which is a cuspidal automorphic representation of k so that

$$L_(s, \pi_v) = \prod_{w|v} L(s, \Pi_w).$$

This special case of functoriality has been established by Arthur and Clozel [1] when K/k is cyclic.

3 The Rankin-Selberg L-function

The unfolding technique of section 1 has wider ramifications. It can be used to establish the analytic continuation and functional equation for a large class of L-functions which fall under the umbrage of Rankin-Selberg theory.

Let $F : H \to \mathbb{C}$ be a Γ-invariant function which is of rapid decay (that is, $F(x + iy) = O(y^N)$ for all $N \geq 1$.) Let

$$C(F, y) = \int_0^1 F(x + iy)dx, \quad y > 0$$

be the constant term of the Fourier expansion. Let

$$L(F, s) = \int_0^\infty C(F, y)y^s \frac{dy}{y^2}$$

be the Mellin transform of $C(F, y)$.

Theorem 3.1 *Let $L^*(F, s) = \pi^{-s}\Gamma(s)\zeta(2s)L(F, s)$. Then, $L(F, s)$ has analytic continuation to the whole complex plane, regular everywhere except for a simple pole at $s = 1$ with residue equal to*

$$\frac{3}{\pi} \int_{\Gamma\backslash H} F(z)dz.$$

The function $L^*(F, s)$ is regular for all $s \neq 0, 1$ and satisfies a functional equation

$$L^*(F, s) = L^*(F, 1 - s).$$

Proof The key idea is to use the decomposition described earlier. We have

$$L(F,s) = \int_0^\infty \int_0^1 F(x+iy) y^{s-2} dxdy.$$

Decomposing the domain of integration as in the "unfolding" technique, this becomes

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma(\Gamma \backslash H)} F(z) y^s \frac{dxdy}{y^2}$$

This can be rewritten as

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash H} F(\gamma z)(\Im(\gamma z))^s \frac{dxdy}{y^2} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash H} F(z)(\Im(\gamma z))^s \frac{dxdy}{y^2}$$

because F is Γ-invariant. Moving the summation inside the integral shows that this is equal to

$$\int_{\Gamma \backslash H} F(z) E(z,s) \frac{dxdy}{y^2}.$$

As $E(z,s)$ has analytic continuation and functional equation, we get the same for $L(F,s)$. □

We now give a few examples on how to apply this theorem.

In the special case that f is a cusp form of weight k, we may apply the above result to $F(z) = y^k |f(z)|^2$ which is easily checked to be Γ-invariant.

A straightforward computation shows that the constant term is

$$y^k \sum_{n=1}^\infty |a_n|^2 e^{-4\pi n y}.$$

The Mellin transform of the constant term is

$$\int_0^\infty y^{k+s} \sum_{n=1}^\infty |a_n|^2 e^{-4\pi n y} \frac{dy}{y^2} = (4\pi)^{-s-k+1} \Gamma(s+k-1) \sum_{n=1}^\infty \frac{|a_n|^2}{n^{s+k-1}}.$$

This proves:

Theorem 3.2 *Let f be a cusp form of weight k for $SL_2(\mathbb{Z})$. If*

$$f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$$

is its Fourier expansion at infinity, then the Dirichlet series

$$\sum_{n=1}^\infty \frac{|a_n|^2}{n^s}$$

has a meromorphic continuation to the whole complex plane. In fact, if

$$\psi(s) = \pi^{-2s-k+1} 2^{-2s} \Gamma(s) \Gamma(s+k-1) \zeta(2s) \sum_{n=1}^\infty \frac{|a_n|^2}{n^s}$$

then $\psi(s)$ extends to a function which is regular for all $s \in \mathbb{C}$ except at $s=1$ where it has a simple pole and residue equal to

$$\frac{3}{\pi} \int_{\Gamma \backslash H} y^k |f(z)|^2 \frac{dxdy}{y^2} = \frac{3}{\pi}(f,f).$$

Moreover, $\psi(s)$ satisfies the functional equation $\psi(s) = \psi(1-s)$.

If we apply the theorem of Chandrasekharan and Narasimhan [5] mentioned in the previous lectures, we deduce that

$$\sum_{n\leq x} |a_n|^2 = \frac{3}{\pi}(f,f)x^k + O(x^{k-2/5})$$

because twice the sum of the coefficients in the Gamma factors (or equivalently the degree in the sense of Selberg) is equal to 4. By taking a single summand in the sum on the left, we deduce that $a_n = O(n^{k/2-1/5})$. The same technique applied to Maass forms gives us $a_n = O(n^{3/10})$.

If we take f and g to be cusp forms (or even with one of them a cusp form), we consider

$$y^k f(z)\overline{g(z)}$$

which is Γ-invariant. If

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

and

$$g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$$

are the respective Fourier expansions at infinity, then the constant term is easily computed to be equal to

$$y^k \sum_{n=1}^{\infty} a_n \overline{b_n} e^{-4\pi n y}.$$

One could also take forms of different weights k_1 and k_2 and consider

$$y^{(k_1+k_2)/2} f(z)\overline{g(z)}.$$

In the end, applying Theorem 3.1 we deduce that

$$\sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^s}.$$

A suitably normalized version of this series (with appropriate Γ-factors, $\zeta(2s)$ and so forth) extends to a function which is regular everywhere except possibly at $s = 1$ where it may have a simple pole with residue equal to

$$\frac{3}{\pi}(f,g).$$

Thus, if f and g are orthogonal to each other, then the normalized series extends to an entire function.

Kronecker's limit formula states that

$$\lim_{s\to 1}\left[E(z,s) - \frac{1}{s-1}\right] = \log(e^{\gamma}/4\pi) - 2\log(\sqrt{y}|\eta(z)|^2)$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, with $q = e^{2\pi i z}$. If f and g are Hecke eigenforms with π_f, π_g being the associated automorphic representations, the Kronecker limit formula allows us to write down an exact formula for the special value $L(1, \pi_f \otimes \pi_g)$.

4 Rankin-Selberg L-functions for GL_n

The general theory for GL_n was initated and developed by Jacquet, Piatetski-Shapiro and Shalika [18], Shahidi [55] and finally completed by Moeglin-Waldspurger [35]. If π_1 and π_2 are cuspidal automorphic representations of GL_m and GL_n of the adele ring over the rationals (say), then the Rankin-Selberg L-function is defined by the Euler product

$$L(s, \pi_1 \otimes \pi_2) = \prod_p L(s, \pi_{1,p} \otimes \pi_{2,p})$$

where for all but finitely many primes p, the Euler factors are given by the formula

$$L(s, \pi_{1,p} \otimes \pi_{2,p}) = \prod_{i,j} \left(1 - \frac{\alpha_{i,p}^{(1)} \alpha_{j,p}^{(2)}}{p^s}\right)^{-1}$$

and

$$L(s, \pi_{r,p}) = \prod_i \left(1 - \frac{\alpha_{i,p}^{(r)}}{p^s}\right)^{-1}$$

for $r = 1, 2$. It is possible to define the Euler factors at all the places so that the final product converges for $\Re(s) > 1$. The completed L-function turns out to be entire unless

$$\pi_2 \simeq \pi_1 \otimes |\det|^{it}$$

for some real number t in which case the function is regular everywhere except at $s = 1 - it$ where it has a simple pole.

LECTURE 4

Oscillations of Fourier Coefficients of Cusp Forms

1 Preliminaries

Last time, we discussed the Rankin-Selberg L-function on $GL_n \times GL_m$ over a number field. This is one of the most powerful methods in the theory that enables us to deduce the meromorphic continuation of the symmetric power L-functions. The general strategy has been first to derive a meromorphic continuation, then to establish holomorphy everywhere and finally by some form of converse theory (again involving some application of the Rankin-Selberg method or the Langlands-Shahidi method) to establish the automorphy of the desired L-function. In this way, one hopes to inductively deduce the holomorphy of the symmetric power L-functions. This strategy has worked so far for only small dimensions and is perhaps illustrated as follows.

Following conventional notation, we shall now denote by $L(s, \pi, r_m)$ the symmetric power L-function attached to a cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ which we had previously denoted by $L_m(s)$. As mentioned earlier, $L(s, \pi \times \pi)$ decomposes as

$$\zeta_F(s)L(s, \pi, r_2)$$

where $\zeta_F(s)$ is the Dedekind zeta function of F. This already gives a meromorphic continuation of $L(s, \pi, r_2)$. When $F = \mathbb{Q}$ and π corresponds to a holomorphic modular form, Shimura [59] had established the holomorphy of $L(s, \pi, r_2)$ by extending the Rankin-Selberg method and making ingenious use of the classical theta function. Gelbart and Jacquet [10] extended this work to all cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ and in addition proved the existence of a cuspidal automorphic representation Π of $GL_3(\mathbb{A}_F)$ such that

$$L(s, \Pi) = L(s, \pi, r_2).$$

This Π is often called the Gelbart-Jacquet lift of π.

But now, we can apply the Rankin-Selberg method to Π. We find,

$$L(s, \Pi \times \Pi) = \zeta_F(s)L(s, \pi, r_2)L(s, \pi, r_4)$$

and thus, we immediately deduce the meromorphic continuation of the 4-th symmetric power L-function.

We could also consider

$$L(s, \pi \times \Pi) = L(s, \pi, r_1)L(s, \pi, r_3)$$

and this gives us the meromorphy of $L(s, \pi, r_3)$. One expects that for each $L(s, \pi, r_m)$ there exists a cuspidal automorphic representation Π_m on $GL_{m+1}(\mathbb{A}_F)$ such that

$$L(s, \Pi_m) = L(s, \pi, r_m).$$

Now by the Clebsch-Gordon formulas for $SU(2,\mathbb{C})$, we have

$$r_m \otimes r_n = r_{m+n} \oplus r_{m+n-2} \oplus \cdots \oplus r_{|m-n|}$$

which is essentially the trigonometric identity

$$\left(\frac{\sin(m+1)\theta}{\sin\theta}\right)\left(\frac{\sin(n+1)\theta}{\sin\theta}\right) = \sum_{j=0}^{n} \frac{\sin(m+n-2j)\theta}{\sin\theta}$$

for $n \leq m$. We leave this as an exercise for the reader.

The essential point is that we may use the Rankin-Selberg method to inductively deduce the meromorphic continuation of the symmetric power L-functions once we have shown the automorphy property. Since we know that each of the symmetric cube and fourth power L-functions are automorphic by the work of Kim-Shahidi [23] and Kim [20], we can inductively obtain the meromorphic continuation of the $L_m(s)$ for $m \leq 8$. Finer analysis of the location of the poles leads to the holomorphy of the $L_m(s)$ for $m \leq 8$ (which uses the Langlands-Shahidi method of Eisenstein series).

2 Rankin's theorem

The discussion below applies equally well to Maass forms. However, for the sake of clarity, we will specialise to the case of classical Hecke eigenforms.

Given a normalized Hecke eigenform of weight k, we let $a(n)$ be the n-th Fourier coefficient and write

$$a(p) = 2p^{(k-1)/2}\cos\theta_p.$$

In general, we have

$$a(p^a) = (p^a)^{(k-1)/2}\frac{\sin(a+1)\theta_p}{\sin\theta_p}$$

as can be easily checked from the recursion for the Hecke operators. Thus, for example, if $a = 1$, we retrieve our formula for $a(p)$ above. Now

$$\frac{\sin(a+1)\theta}{\sin\theta} = \frac{e^{i(a+1)\theta} - e^{-i(a+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{x^{a+1} - y^{a+1}}{x-y}$$

with $x = e^{i\theta}, y = e^{-i\theta}$, so we see from

$$\frac{x^{a+1} - y^{a+1}}{x-y} = x^a + x^{a-1}y + \cdots + y^a$$

that

$$\left|\frac{\sin(a+1)\theta}{\sin\theta}\right| \leq a+1.$$

Therefore,

$$|a(n)| \leq n^{(k-1)/2} d(n)$$

where $d(n)$ is the number of divisors of n. The maximal order of $d(n)$ is easily determined (see for example, [12]) and we have

$$a(n) = O\left(n^{(k-1)/2}\exp\left(\frac{c\log n}{\log\log n}\right)\right)$$

for some suitable constant $c > 0$.

In 1973, Rankin [45] investigated if this is the optimal error term. He proved that

$$\limsup_{n\to\infty} \frac{|a(n)|}{n^{(k-1)/2}} = +\infty.$$

In the same paper, Rankin indicated that if the Sato-Tate conjecture is true, then

$$a(n) = \Omega_{\pm}\left(n^{(k-1)/2}\exp\left(\frac{c\log n}{\log\log n}\right)\right)$$

for some $c > 0$. This Ω-estimate was finally proved unconditionally in [41]. We briefly indicate Rankin's argument. By the Sato-Tate conjecture, we have

$$\#\{p \le x : \theta_p \in [-\pi/6, \pi/6]\} \ge c\pi(x).$$

Put

$$N = \prod_{\substack{p\le x \\ \theta_p\in[-\pi/6,\pi/6]}} p.$$

Then

$$a(N) = \prod_{\substack{p\le x \\ \theta_p\in[-\pi/6,\pi/6]}} a(p).$$

Because

$$|a(p)| = 2p^{(k-1)/2}|\cos\theta_p| \ge \sqrt{3}p^{(k-1)/2}$$

we obtain

$$|a(N)| \ge N^{(k-1)/2}(\sqrt{3})^{c\pi(x)}.$$

Now, by partial summation

$$\log N = \sum_{\substack{p\le x \\ \theta_p\in[-\pi/6,\pi/6]}} \log p \ge c_1 x.$$

Also, by Chebycheff's estimate

$$\log N \le c_2 x.$$

In any case

$$|a(N)| \ge N^{(k-1)/2}(\sqrt{3})^{c_0 \log N/\log\log N}$$

and the omega theorem is deduced from this.

But since we don't have the Sato-Tate conjecture in its entirety until all the symmetric power L-functions are shown to be entire, (no pun intended) it makes sense to ask how much of the Sato-Tate conjecture can be proved if we only had analyticity of $L_m(s)$ for $m \le R$ (say). For instance, can we aim for Chebycheff type estimates for the Sato-Tate problem based on only partial information. The goal of this lecture is to indicate how we may deduce the following.

Theorem 4.1 *Suppose that $L_r(s)$ extends to an analytic function for $\Re(s) \ge 1/2$ for all $r \le 2m+2$. Then, each of the statements*

1. *for any $\delta > 0$, $-\delta < 2\cos\theta_p < \frac{2}{\delta(m+2)}$;*
2. *for any $\epsilon > 0$,*

$$|2\cos\theta_p| > \sqrt{\frac{4m+2}{m+2}} - \epsilon;$$

3. *for any* $\epsilon > 0$, $2\cos\theta_p > \beta_m - \epsilon$ *where*

$$\beta_m = \left\{ \frac{1}{4(m+2)} \binom{2m+2}{m+1} \right\}^{\frac{1}{2m+1}};$$

holds for a positive density of primes.

Corollary 4.2 *Setting* $\delta = \sqrt{2/(m+2)}$ *in (1), we deduce that there is a positive density of primes p satisfying*

$$-\sqrt{\frac{2}{m+2}} < 2\cos\theta_p < \sqrt{\frac{2}{m+2}}.$$

Putting $m = 1$ in (2), we deduce

Corollary 4.3 *For a positive density of primes p, we have*

$$|a(p)| \geq (\sqrt{2} - \epsilon)p^{(k-1)/2}.$$

This last result is what we need to obtain Rankin's oscillation theorem without the Sato-Tate conjecture. For this, we need the analyticity for $\Re(s) \geq 1/2$ for each of $L_r(s)$, $r \leq 4$.

In 1981, Shahidi [55] proved that $L_3(s)$ and $L_4(s)$ are analytic in this region. Recently Kim and Shahidi [23] established that these are in fact automorphic L-functions and hence entire. This is more information than we need and it is quite possible that this can be used to refine our results.

3 A review of symmetric power L-functions

Let us look at

$$L_r(s) = \prod_p \prod_{j=0}^{r} \left(1 - \frac{\alpha_p^{r-j}\beta_p^j}{p^s}\right)^{-1}.$$

Now $\alpha_p = e^{i\theta_p}, \beta_p = e^{-i\theta_p}$ so that the Euler factor is

$$\prod_{j=0}^{r} \left(1 - \frac{e^{i(r-2j)\theta_p}}{p^s}\right)^{-1}.$$

If $L_r(s)$ is analytic for $\Re(s) \geq 1$, and non-vanishing there, we may apply the Tauberian theorem to deduce

$$\sum_{p\leq x} \left(\sum_{j=0}^{p} e^{i(r-2j)\theta_p} \right) = o(x/\log x).$$

By an easy exercise,

$$\frac{\sin(r+1)\theta}{\sin\theta} = \sum_{j=0}^{r} e^{i(r-2j)\theta}$$

so that for $r \geq 1$,

$$\sum_{p\leq x} \frac{\sin(r+1)\theta_p}{\sin\theta_p} = o(x/\log x).$$

Let

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

and $T_n(\cos\theta) = \cos n\theta$ be the Chebycheff polynomials of the first and second kind respectively.

We have the identity

$$2T_n(x) = U_n(x) - U_{n-2}(x).$$

Thus, for $n = 2$,

$$2\sum_{p\le x} T_2(\cos\theta_p) = \sum_{p\le x} U_2(\cos\theta_p) - \pi(x)$$

which implies

$$\sum_{p\le x} T_2(\cos\theta_p) = (-\frac{1}{2} + o(1))\pi(x).$$

We also have

$$\sum_{p\le x} T_1(\cos\theta_p) = o(x/\log x).$$

Now we can write powers of the cosine function using Chebycheff polynomials of the second kind:

$$(2\cos\theta)^r = 2\sum_{k=0}^{r'} \binom{r}{k} T_{r-2k}(\cos\theta), \quad r' = [r/2].$$

From this identity, we deduce

$$\sum_{p\le x}(2\cos\theta_p)^r = 2\sum_{k=0}^{r'} \binom{r}{k} \sum_{p\le x} T_{r-2k}(\cos\theta_p).$$

Each of the inner sums is $o(x/\log x)$ unless $r - 2k = 0$ or 2 in which case it is $\pi(x)$ or $-\pi(x)/2$ respectively. Hence, if r is not even, the sum is $o(x/\log x)$. The final result is

$$\sum_{p\le x}(2\cos\theta_p)^{2r} = \left(-\binom{2r}{r-1} + \binom{2r}{r}\right)(1+o(1))\frac{x}{\log x}.$$

The term inside the brackets is

$$\frac{1}{r+1}\binom{2r}{r}.$$

We can state the final result as:

Theorem 4.4

$$\sum_{p\le x}(2\cos\theta_p)^{2r} = \frac{1}{r+1}\binom{2r}{r}(1+o(1))\frac{x}{\log x}$$

as x tends to infinity.

For example, when $r = 1$, we get

$$\sum_{p\le x}(2\cos\theta_p)^2 = (1+o(1))\frac{x}{\log x}$$

and for $r = 2$,

$$\sum_{p\le x}(2\cos\theta_p)^4 = (2+o(1))\frac{x}{\log x}.$$

This last result immediately gives that for a positive proportion of primes, we have

$$|2\cos\theta_p| \ge 2^{1/4} - \epsilon$$

which is greater than 1. From this result, we may deduce the Ω-result stated earlier. See [41] for further details.

4 Proof of Theorem 4.1

To prove Theorem 4.1, we need to obtain finer information about sign changes and so forth, and a slightly subtler analysis is needed.

We will need the following combinatorial identities.

Lemma 4.5

$$\sum_{j=0}^{r}(-1)^j \binom{r}{j} \binom{2j}{j} \frac{2^{-2j}}{j+1} = 2^{-2r-1} \binom{2r+2}{r+1} \tag{1}$$

$$\sum_{j=0}^{r}(-1)^j \binom{r}{j} \binom{2j+2}{j+1} \frac{2^{-2j}}{j+2} = \frac{2^{-2r}}{r+2} \binom{2r+2}{r+1}. \tag{2}$$

Now consider the polynomial

$$P_m(x) = (x^2-4)^m(x-a)(x-b)$$

where a and b will be chosen later. Now

$$P_m(x) = (x^2-(a+b)x+ab)\sum_{j=0}^{m} \binom{m}{j} x^{2j}(-1)^{m-j}4^{m-j}$$

so that (after some calculation) we find

$$\frac{\log x}{x} \sum_{p \le x} P_m(2\cos\theta_p) \sim \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j}4^{m-j} \left(\frac{1}{j+2} \binom{2j+2}{j+1} + \frac{ab}{j+1} \binom{2j}{j} \right)$$

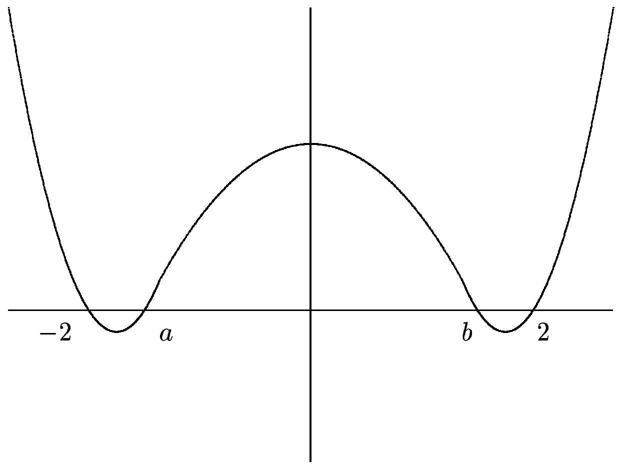
which by the lemma is

$$\sim (-1)^m \binom{2m+2}{m+1} \left(\frac{ab}{2} + \frac{1}{m+2} \right).$$

We conclude that

$$\sum_{p \le x} P_m(2\cos\theta_p) \sim (-1)^m \binom{2m+2}{m+1} \left(\frac{ab}{2} + \frac{1}{m+2} \right) \frac{x}{\log x}.$$

Now, we examine the graph of $P_m(x)$.



The graph of $P_m(x)$

Choosing $a = -\delta$, b so that $ab > -2/(m+2)$ if m is even and $ab < -2/(m+2)$ if m is odd, we deduce that

$$\sum_{p\le x} P_m(2\cos\theta_p) \gg \frac{x}{\log x}.$$

This means that a positive proportion of primes will have

$$a < 2\cos\theta_p < b$$

so we get

$$-\delta < 2\cos\theta_p < \frac{2}{\delta(m+2)}$$

as stated in Theorem 4.1.

The remaining part (2) of the Theorem are obtained by using the polynomial

$$Q_m(x) = x^{2m}(x^2-\gamma)$$

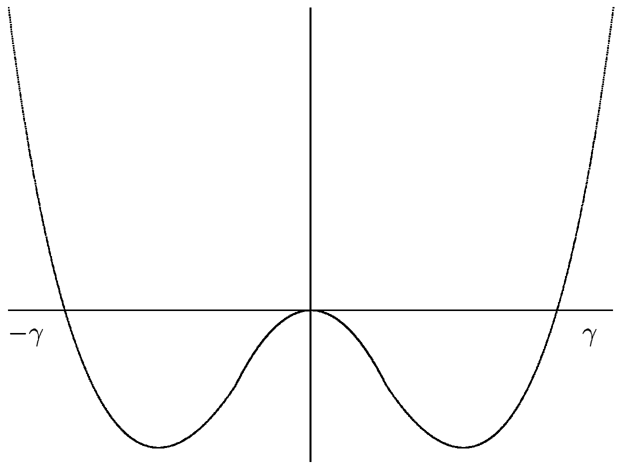
where γ is to be chosen. As before, we deduce

$$\sum_{p\le x} Q_m(2\cos\theta_p) \sim \left(\frac{1}{m+2}\binom{2m+2}{m+1} - \frac{\gamma}{m+1}\binom{2m}{m}\right)\frac{x}{\log x}$$

as x tends to infinity. Again, examining the graph of $Q_m(x)$, we see that if

$$\gamma = \frac{m+1}{m+2}\binom{2m+2}{m+1}\binom{2m}{m}^{-1} - \epsilon = \frac{2(2m+1)}{m+2} - \epsilon$$

we get (2).



The graph of $Q_m(x)$

Finally,

$$\sum_{p\leq x} |2\cos\theta_p|^{2m+1} \geq \frac{1}{2}\sum_{p\leq x}(2\cos\theta_p)^{2m+2}$$

which implies that

$$\sum_{p\leq x,\, a_p>0} (2\cos\theta_p)^{2m+1} \gtrsim \frac{1}{4(m+2)}\binom{2m+2}{m+1}\frac{x}{\log x}.$$

Thus, for a positive proportion of primes

$$2\cos\theta_p > \left\{\frac{1}{4(m+2)}\binom{2m+2}{m+1}\right\}^{\frac{1}{2m+1}} - \epsilon.$$

By choosing better polynomials, Rankin [46], Serre and Shahidi [58] have obtained refined results. Most notable is Rankin's result[46] that for some $\delta > 0$, we have

$$\sum_{n\leq x} |a_n/n^{(k-1)/2}| \ll \frac{x}{(\log x)^\delta}.$$

Here is a sketch of Rankin's argument. Let

$$b_n = a_n/n^{(k-1)/2}.$$

For each r, define the series

$$\psi_r(s) = \prod_p (1 - 2(\cos r\theta_p)p^{-s} + p^{-2s})^{-1}.$$

Then,

$$\zeta(s)\psi_2(s) = L_2(s)$$

and

$$\zeta(s)\psi_2(s)\psi_4(s) = L_4(s),$$

as can be easily checked by comparing the Euler factors of both sides.

By Gelbart-Jacquet [10] (for $L_2(s)$) and Shahidi [55] (for $L_4(s)$) we see that

$$\zeta(s)\psi_2(s)$$

and

$$\zeta(s)\psi_2(s)\psi_4(s)$$

are holomorphic and non-vanishing for $\Re(s) \geq 1$.

Rankin [46] shows that there are functions K, L, M in β satisfying

$$K - L = F(\beta) + 1$$

such that if

$$u^+(\theta) = K + 2L\cos 2\theta + 2M\cos 4\theta$$

then

$$|2\cos\theta|^{2\beta} \leq u^+(\theta)$$

and

$$F(\beta) = \frac{2^{\beta-1}}{5}(2^\beta + 3^{2-\beta}) - 1.$$

We consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a^+(n)}{n^s} = \prod_p A_p^+(s)$$

where

$$A_p^+(s) = 1 + u^+(\theta_p)p^{-s} + \sum_{v=2}^{\infty}(v+1)^{2\beta}p^{-vs}$$

so that for all real values of s,

$$\sum_{n=1}^{\infty} \frac{|b_n|^{2\beta}}{n^s} \leq \sum_{n=1}^{\infty} \frac{a^+(n)}{n^s} = \zeta(s)^K \psi_2(s)^L \psi_4(s)^M H_3(s)$$

where $H_3(s)$ is holomorphic and on-zero for $\Re(s) > 1/2$. By an extended version of the Tauberian theorem (due to Delange), we obtain

$$\sum_{n \leq x} a^+(n) \sim cx(\log x)^{K-L-1}$$

with $c \neq 0$, and if $K - L \leq 1$ (note that there is a typo on p. 175, line -13 of [58]).

We now use the fact that $K - L = F(\beta) + 1$ and for $\beta = 1/2$, $F(1/2) = \frac{\sqrt{2}}{5}(\sqrt{2} + 3\sqrt{3}) - 1 < 0$ as is easily checked. This completes the proof (sketch) of Rankin's theorem.

Rankin's theorem was used by Murty-Murty[40] in proving a crucial non-vanishing theorem which was an essential ingredient for Kolyvagin's theorem about finiteness of Tate-Shafarevich groups of modular elliptic curves with Mordell-Weil rank ≤ 1.

LECTURE 5

Poincaré Series

1 Poincaré series for $SL_2(\mathbb{Z})$

The Poincaré series for $SL_2(\mathbb{Z})$ are defined by

$$G_r(z) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k} e^{2\pi i r \frac{az+b}{cz+d}}$$

where a, b are any integers such that $ad - bc = 1$. Observe that if $r = 0$, this reduces to the classical Eisenstein series $E_k(z)$ (upto a constant). Thus, the Poincaré series are to be viewed as generalisations of Eisenstein series. It is easy to see that the inner summand does not depend on the choice of a solution. Indeed, by the Euclidean algorithm, any other solution for (a, b) has the form $(a + tc, b + td)$ and

$$\frac{(a+tc)z + (b+td)}{cz+d} = \frac{az+b}{cz+d} + t, \quad t \in \mathbb{Z}$$

so that

$$e^{2\pi i r\left(\frac{az+b}{cz+d}+t\right)} = e^{2\pi i r\left(\frac{az+b}{cz+d}\right)}.$$

We can rewrite the series in a more invariant form by setting

$$j(\gamma, z) = cz + d, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and then

$$G_r(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} e^{2\pi i r(\gamma z)}.$$

The important thing to note is that $G_r(z)$ is a modular form of weight k. To see this, let $\delta \in \Gamma = SL_2(\mathbb{Z})$. Then,

$$G_r(\delta z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \delta z)^{-k} e^{2\pi r(\gamma \delta z)}.$$

Now, we have the so-called cocycle relation:

$$j(\gamma\delta, z) = j(\gamma, \delta z) j(\delta, z)$$

as is easily verified, so that

$$j(\gamma, \delta z) = \frac{j(\gamma\delta, z)}{j(\delta, z)}$$

and

$$\begin{aligned} G_r(\delta z) &= j(\delta, z)^k \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma\delta, z)^{-k} e^{2\pi i r(\gamma\delta z)} \\ &= j(\delta, z)^k G_r(z). \end{aligned}$$

Holomorphy is easy to verify using standard tests of complex analysis. In addition, $G_r(i\infty) = 0$ if $r \geq 1$. We conclude that for every $r \geq 1$, $G_r(z)$ is a cusp form of weight k. Thus, Poincaré series give explicit constructions of cusp forms. For a detailed treatment of this theory, we refer the reader to Rankin's book [44] (especially Chapter 5).

Now let f be any cusp form of weight k. We would like to compute the inner product (f, G_r).

First observe that $e^{2\pi iz} = e^{2\pi ix} \cdot e^{-2\pi y}$ so that

$$\overline{e^{2\pi iz}} = e^{-2\pi ix} \cdot e^{-2\pi y} = e^{-2\pi i(\overline{z})}.$$

Thus,

$$\begin{aligned}
(f, G_r) &= \int_{\Gamma\backslash H} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(z) e^{-2\pi i r\overline{\gamma z}} (\overline{cz+d})^{-k} y^k \frac{dxdy}{y^2} \\
&= \int_{\Gamma\backslash H} \sum_{\Gamma_\infty\backslash\Gamma} (cz+d)^k f(z) e^{-2\pi i r(\overline{\gamma z})} \frac{y^k}{|cz+d|^{2k}} \frac{dxdy}{y^2} \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash H} f(\gamma z) \Im(\gamma z)^k e^{-2\pi i r(\overline{\gamma z})} \frac{dxdy}{y^2} \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma(\Gamma\backslash H)} f(z) \Im(z)^k e^{-2\pi i r\overline{z}} \frac{dxdy}{y^2} \\
&= \int_0^\infty \int_0^1 f(x+iy) y^k e^{-2\pi i r x} e^{-2\pi r y} \frac{dxdy}{y^2}.
\end{aligned}$$

Now from the Fourier expansion of $f(z)$:

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n x} \cdot e^{-2\pi n y}$$

we see that the x-integral picks up the r-th Fourier coefficient. Thus,

$$(f, G_r) = a_r \int_0^\infty e^{-4\pi r y} y^{k-2} dy.$$

By setting $4\pi r y = t$ in the integrand and simplifying, we deduce

Theorem 5.1 *Let f be any cusp form of weight k for Γ. Then,*

$$(f, G_r) = \frac{\Gamma(k-1) a_r}{(4\pi r)^{k-1}}.$$

An important corollary is:

Corollary 5.2 *Every cusp form is a finite linear combination of Poincaré series $G_r(z)$.*

Proof The set of Poincaré series spans a closed subspace in the space of cusp forms. If f is a cusp form not in this space, all of its Fourier coefficients must vanish by the previous theorem. Thus, the orthogonal complement is zero. □

As an example, consider the case $k = 12$. Each of the $G_r(z)$ is a cusp form of weight 12. But any cusp form of weight 12 must be a constant multiple of Δ, Ramanujan's cusp form. Thus,

$$G_r(z) = c_r \Delta(z).$$

What is c_r? By the above theorem,

$$(\Delta, G_r) = \frac{\Gamma(11)\tau(r)}{(4\pi r)^{11}} = c_r(\Delta, \Delta).$$

Hence,

$$\tau(r) = \frac{(4\pi r)^{11}}{10!} \int_{\Gamma\backslash H} y^{12}\Delta(z)\overline{G_r(z)}\frac{dxdy}{y^2}.$$

2 Fourier coefficients and Kloosterman sums

Emanating largely from the work of Petersson [43] in the 1930's and Selberg [51], an explicit formula can be given for the Fourier coefficients of $G_r(z)$. This striking formula involves the Kloosterman sums and their appearance has opened new connections to the Selberg eigenvalue conjecture as well as applications to classical questions of analytic number theory. We now derive this remarkable formula.

We begin by writing

$$G_r(z) = \sum_{n=1}^{\infty} g_{rn} e^{2\pi i n z}.$$

Then,

$$g_{rn} = \int_0^1 G_r(x)e^{-2\pi i n x}.$$

More precisely, for reasons of convergence, we should consider

$$\int_{i\delta}^{1+i\delta} G_r(z)e^{-2\pi i n z}$$

with $\delta > 0$, but we leave this technical modification to the reader. We have

$$g_{rn} = \frac{1}{2}\sum_{(c,d)=1}\int_0^1 (cx+d)^{-k}e^{2\pi i r\left(\frac{ax+b}{cx+d}\right)-2\pi i n x}dx.$$

Put $cx + d = t$. The argument in the exponential becomes

$$\frac{r}{t}\left(\frac{a}{c}(t-d)+b\right) - \frac{n}{c}(t-d) = \frac{nd+ar}{c} - \frac{nt}{c} - \frac{r}{tc}$$

since $ad - bc = 1$. Thus,

$$g_{rn} = \frac{1}{2}\sum_{c\neq 0}\frac{1}{c}\sum_{\substack{d \pmod c\\ ad\equiv 1 \pmod c}} e^{\frac{2\pi i}{c}(nd+ar)}\int_{-\infty}^{\infty} t^{-k}e^{-\frac{2\pi i}{c}\left(\frac{r}{t}+nt\right)}dt$$

because

$$\sum_{(c,d)=1}\int_0^1 t^{-k}e^{\frac{2\pi i}{c}(nd+ar-nt)}e^{-\frac{2\pi i r}{t}}dt$$

depends only on $d \pmod c$. Writing d as $d_0 + (m+1)c$ with varying m, we transform the integral from 0 to 1 into an integral from $-\infty$ to ∞. This integral turns out to be a Bessel function:

$$\int_{-\infty+ci}^{\infty+ci} t^{-k}\exp\left(-\frac{2\pi i}{c}\left(\frac{r}{t}+nt\right)\right)dt = 2\pi(n/r)^{(k-1)/2}J_{k-1}(4\pi\sqrt{rn}/c)$$

where

$$J_k(z) = \frac{1}{2\pi i}\int_C t^{-k-1}e^{\frac{z}{2}(t-1/t)}dt$$

where C is the unit circle. The sum

$$S(r,n,c) := \sum_{\substack{d (\mathrm{mod}\ c) \\ ad\equiv 1 (\mathrm{mod}\ c)}} e^{\frac{2\pi i}{c}(nd+ar)}$$

is called a Kloosterman sum. Using this notation, we obtain the beautiful formula due to Petersson:

$$g_{rn} = (n/r)^{(k-1)/2}\left\{\delta_{rn} + \pi\sum_{c=1}^{\infty}\frac{S(r,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right)\right\}$$

where δ_{rn} denotes the Kronecker delta function.

We have already noted that the Poincaré series span the space of cusp forms. Thus, to prove the Ramanujan conjecture, it suffices to show that

$$g_{rn} = O(n^{\frac{k-1}{2}+\epsilon})$$

for every r. This is tantamount to showing that the expression in parentheses in the above sum is $O(n^{\epsilon})$.

Selberg, using this expression and Weil's estimate for Kloosterman sums:

$$|S(r,n,c)| \le d(c)c^{1/2}(r,n,c)^{1/2}$$

as well as the bound

$$J_{k-1}(x) \le A\min(x^{k-1}, x^{-1/2})$$

obtained that

$$g_{rn} = O(n^{k/2-1/4+\epsilon}).$$

Note that this is better than what we obtained earlier by the Rankin-Selberg method. Since the estimate was obtained crudely, Selberg felt that there must be cancellation among the Kloosterman sums. This led him to formulate the following conjecture (which was also arrived at independently by Linnik):

Conjecture (Selberg-Linnik)

$$G(x) := \sum_{c\le x}\frac{S(r,n,c)}{c} = O(x^{\epsilon})$$

for $x \ge \gcd(r,n)^{1/2+\epsilon}$ *for any* $\epsilon > 0$.

In his 1965 paper, Selberg stated that this would lead to a proof of the Ramanujan conjecture (for Maass forms as well) but did not indicate a proof. We will indicate below how such a proof can be obtained for the full modular group. The argument is adapted from the author's [41].

Let us first observe that Weil's estimate for Kloosterman sums leads to the estimate

$$G(x) = O(x^{1/2}\log x)$$

for $(r,n) = 1$. Kuznetsov[26] proved that $G(x) = O(x^{1/6+\epsilon})$, but the O-constant depends on r, n so it is not applicable to the estimation of the Fourier coefficients. Let

$$H(x) := \sum_{c\le x} S(r,n,c).$$

By partial summation, the Selberg-Linnik conjecture is equivalent to

$$H(x) = O(x^{1+\epsilon}).$$

We begin by considering

$$\sum_{c>\sqrt{n}} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) = \sum_{c>\sqrt{n}} G(c) \left\{ J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c+1}\right) - J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) \right\}$$

by partial summation. By the mean value theorem, the expression in parentheses is

$$\frac{4\pi\sqrt{rn}}{c(c+1)} J'_{k-1}(\xi_c)$$

for some $\xi_c \in (4\pi\sqrt{rn}/(c+1), 4\pi\sqrt{rn}/c)$. Using the estimate

$$J'_{k-1}(x) \ll x^{-1/2}$$

we get

$$\sum_{c>\sqrt{rn}} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) \ll n^{1/4} \sum_{c>\sqrt{n}} \frac{|G(c)|}{c^{3/2}} \ll n^{\epsilon},$$

by the Selberg-Linnik conjecture. Thus, we need only consider

$$\sum_{c\leq\sqrt{n}} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right).$$

To estimate this, we apply an inductive argument. As there are no cusp forms of weight 10, we have

$$\sum_{c\leq\sqrt{n}} \frac{S(r,n,c)}{c} J_9\left(\frac{4\pi\sqrt{rn}}{c}\right) = O(n^{\epsilon}).$$

So, if for example, we were trying to establish the conjecture for $k = 12$, then it suffices to estimate for $k = 10$ the quantity

$$\sum_{c\leq\sqrt{n}} \frac{S(r,n,c)}{c} \left\{ J_{k+1}\left(\frac{4\pi\sqrt{rn}}{c}\right) + J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) \right\}.$$

By the familiar identity

$$\frac{2kJ_k(x)}{x} = J_{k+1}(x) + J_{k-1}(x)$$

it suffices to estimate

$$\frac{1}{\sqrt{n}} \sum_{c\leq\sqrt{n}} S(r,n,c) J_k\left(\frac{4\pi\sqrt{rn}}{c}\right).$$

Again, by partial summation, we may write this as

$$\frac{1}{\sqrt{n}} \sum_{c\leq\sqrt{n}} H(c) \left\{ J_k\left(\frac{4\pi\sqrt{rn}}{c+1}\right) - J_k\left(\frac{4\pi\sqrt{rn}}{c}\right) \right\}.$$

Again, the expression in the brackets is

$$\frac{4\pi\sqrt{rn}}{c(c+1)} J'_k(\xi_c).$$

Using the estimate

$$J'_k(x) \ll x^{-1/2}$$

as before, and the fact that $H(c) = O(c^{1+\epsilon})$, we deduce a final estimate of $O(n^{\epsilon})$ as desired. This completes the proof of the fact that the Selberg-Linnik conjecture implies the Ramanujan conjecture (for the full modular group). A similar argument can be applied to higher levels. However, the non-existence of cusp forms of small weight is not guaranteed. In this case, we exploit the fact that we know the Ramanujan conjecture in the weight two case (a result due to Eichler and Shimura).

3 The Kloosterman-Selberg zeta function

In order to gain more insight into the Selberg-Linnik conjecture, we will consider (with Selberg [52]) the series

$$Z(r,n,s) = \sum_{c\neq 0} \frac{S(r,n,c)}{|c|^{2s}}.$$

To study this series, Selberg [52] considers the cognate Poincaré series

$$U_n(z,s) = \sum_{\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s e^{2\pi i n\gamma z}.$$

Clearly, U_n is Γ-invariant. Moreover, if

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

then

$$\Delta U_n(z,s) = s(1-s)U_n(z,s) + 4\pi n U_n(z,s+1)s.$$

As we will show next time, the Fourier expansion of $U_n(z,s)$ contains $Z(r,n,s)$ in it. This allows us to relate the eigenvalues of Δ with the abscissa of convergence of $Z(r,n,s)$. More precisely,

$$U_n(z,s) = \sum_{m=-\infty}^{\infty} B_n(m,y,s)e^{2\pi i m x}$$

where

$$\begin{aligned} B_n(m,y,s) &= \delta_{nm}y^s e^{-2\pi n y} + \frac{1}{2}\sum_{c\neq 0}\frac{S(m,n,c)}{|c|^{2s}}y^{1-s} \\ &\times \int_{-\infty}^{\infty} \exp\left(-2\pi i m y v - \frac{2\pi n}{c^2 y(1-iv)}\right)\frac{dv}{(1+v^2)^s}. \end{aligned}$$

It then turns out that

$$\begin{aligned} (2\pi\sqrt{nm})^{2s-1}\sum_{c=1}^{\infty}\frac{S(n,m,c)}{|c|^{2s}} &= \frac{\sin\pi s}{2}\sum_{j=1}^{\infty}\frac{a_j(n)\overline{a_j(m)}}{\cosh\pi r_j}\Gamma(s-\frac{1}{2}+ir_j)\Gamma(s-\frac{1}{2}-ir_j) \\ &-\frac{\delta_{nm}}{2\pi}\frac{\Gamma(s)}{\Gamma(1-s)} + \frac{1}{\pi}\int_{-\infty}^{\infty}(n/m)^{ir}\sigma_{2ir}(m)\sigma_{-2ir}(m)\frac{h(r,s)}{|\zeta(1+2ir)|^2}dr \end{aligned}$$

where

$$h(r,s) = \frac{\sin\pi s}{2}\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)$$

and the $a_j(n)$'s are the Fourier coefficients of the Maass form corresponding to the eigenvalue $\lambda_j = 1/4+r_j^2$. This remarkable formula establishes a striking relationship between the eigenvalues λ_j and the Kloosterman-Selberg zeta function.

LECTURE 6

Kloosterman sums and Selberg's Conjecture

1 Petersson's formula

Recall that we defined the Poincaré series $G_r(z)$ for $SL_2(\mathbb{Z})$ by

$$G_r(z) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k} e^{2\pi i r\left(\frac{az+b}{cz+d}\right)}.$$

For $r = 0$, we retrieve the classical modular form of weight k.

We have the important:

Theorem 6.1 *If f is a cusp form with Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},$$

then

$$(f, G_r) = \frac{\Gamma(k-1) a_r}{(4\pi r)^{k-1}}.$$

In particular, every cusp form can be written as a finite sum of Poincaré series.

A straightforward calculation allows us to write down an explicit formula for the n-th Fourier coefficient of $G_r(z)$. More precisely, if

$$G_r(z) = \sum_{n=1}^{\infty} g_{rn} e^{2\pi i n z}$$

then

$$g_{rn} = (n/r)^{(k-1)/2} \left\{ \delta_{rn} + \pi \sum_{c=1}^{\infty} \frac{S(r,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right) \right\}.$$

If in Petersson's formula, we put $f = G_t$, we obtain a "quasi"-orthogonality relation among the Fourier coefficients of the Poincaré series. We state this as:

Theorem 6.2 *Let $f_1, ..., f_r$ be an orthonormal basis of the space of cusp forms of weight k on Γ. Suppose*

$$f_i(z) = \sum_{n=1}^{\infty} a_{f_i}(n) e^{2\pi i n z}.$$

Then

$$\frac{\Gamma(k-1)}{4\pi (rt)^{(k-1)/2}} \sum_{i=1}^{r} a_{f_i}(r) a_{f_i}(t) = \delta_{rt} + \pi \sum_{c>0} \frac{S(t,r,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{rt}}{c}\right).$$

Proof We write

$$G_t = \sum_i (f_i, G_t) f_i.$$

Using Petersson's formula for the inner product and comparing the n-th coefficient of both sides gives us the result. □

Iwaniec interprets this theorem as the $GL(2)$-analogue of the classical orthogonality relations for Dirichlet characters. Thus, from this viewpoint, coefficients of Poincaré series are analogous to Dirichlet characters and this analogy suggests a universe of exploration especially from the standpoint of analytic number theory. For example, following the treatment of Davenport one can successfully develop a 'spectral' version of the large sieve. We refer the reader to the finer expositions of [15] and [38] for further details. These ideas first appeared in [52].

2 Selberg's theorem

Last time, we introduced the Kloosterman-Selberg zeta function:

$$Z(r,n,s) = \sum_{c \neq 0} \frac{S(r,n,c)}{|c|^{2s}}$$

where

$$S(r,n,c) = \sum_{\substack{d (\mathrm{mod}\ c) \\ ad \equiv 1 (\mathrm{mod}\ c)}} e^{\frac{2\pi i}{c}(ar+nd)}$$

is the Kloosterman sum. The remarkable connection between Selberg's eigenvalue conjecture and $Z(r,n,s)$ is given by the following theorem.

Theorem 6.3 (Selberg) *Let $\sigma_0 > 1/2$. $Z(r,n,s)$ admits analytic continuation to $\Re(s) > \sigma_0$ if and only if $\lambda_1 \geq \sigma_0(1-\sigma_0)$.*

The trivial bound $S(r,n,c) = O(c)$ gives that $Z(r,n,s)$ extends to an analytic function for $\Re(s) > 1$ and Weil's estimate gives it for $\Re(s) > 3/4$. The latter result implies by the theorem that $\lambda_1 \geq 3/16$. This is how Selberg obtains his result for general congruence subgroups.

However, in the full modular case, one can show $\lambda_1 \geq 1/4$ by a simple elementary argument (due to Roelcke). This gives an analytic continuation of $Z(r,n,s)$ to $\Re(s) > 1/2$. Of course, if we admit the Selberg-Linnik conjecture, we immediately get $\lambda_1 \geq 1/4$ by the theorem.

Theorem 6.4 *For the full modular group, we have*

$$\lambda_1 \geq \frac{3\pi^2}{2}.$$

Proof Let $u(z)$ be a Maass form with eigenvalue λ. Without loss of generality, we may suppose that $(u,u) = 1$. Thus,

$$\lambda = (\Delta u, u) = \int_{\mathcal{F}} (\Delta u)\overline{u} \frac{dxdy}{y^2}.$$

Let

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, it is easy to see that if $\mathcal{F}_1 = S(\mathcal{F})$, we have

$$2\lambda = \int_{\mathcal{F}\cup\mathcal{F}_1} (\Delta u)\overline{u}\frac{dxdy}{y^2}.$$

By Stokes theorem, this is

$$\int_{\mathcal{F}\cup\mathcal{F}_1} \left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \right) dxdy$$

which is

$$\geq \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{-1/2}^{1/2} \left|\frac{\partial u}{\partial x}\right|^2 dxdy.$$

Writing

$$u(x+iy) = \sum_{n\neq 0} a_n(y)e^{2\pi inx}$$

we get

$$\frac{\partial u}{\partial x} = \sum_{n\neq 0} a_n(y)2\pi ine^{2\pi inx}$$

so that

$$\int_{-1/2}^{1/2} \left|\frac{\partial u}{\partial x}\right|^2 dx = \sum_{n\neq 0} |a_n(y)|^2(2\pi n)^2.$$

Putting this in our previous formula, we get

$$\begin{aligned}
2\lambda &\geq \int_{\sqrt{3}/2}^{\infty} \sum_{n\neq 0} |2\pi n a_n(y)|^2 dy \\
&\geq \frac{3}{4}\int_{\sqrt{3}/2}^{\infty} \sum_{n\neq 0} (2\pi n a_n(y))^2 \frac{dy}{y^2} \\
&\geq 3\pi^2 \int_{\sqrt{3/2}}^{\infty} \int_{-1/2}^{1/2} \sum_{n\neq 0} |a_n(y)|^2 \frac{dxdy}{y^2} \\
&\geq 3\pi^2(u,u) = 3\pi^2.
\end{aligned}$$

This completes the proof. □

3 The Selberg-Linnik conjecture

I want to now sketch a proof that the abscissa of convergence of the series

$$\sum_{c=1}^{\infty} \frac{S(m,n,c)}{|c|^{2s}}$$

is related to the eigenvalue λ_1. Last time, we introduced

$$U_m(z,s) = \sum_{\Gamma_\infty\backslash\Gamma} (\Im(\gamma z))^s e^{2\pi im\gamma z}.$$

It is easy to see that

$$\{\Delta + s(1-s)\}U_m(z,s) = -4ms\pi U_m(z,s+1)$$

and this relation can be used to meromorphically continue $U_m(z,s)$ to the entire complex plane. See [38] for details. Let now ψ_j be a Maass cusp form. The analog of Petersson's formula is

$$(U_m(\cdot,s),\psi_j) = \frac{\overline{a_j(m)}\sqrt{\pi}}{(4\pi m)^{s-1/2}}\frac{\Gamma(s-1/2+ir_j)\Gamma(s-1/2-ir_j)}{\Gamma(s)}$$

where $\lambda_j = 1/4 + r_j^2$ is the corresponding eigenvalue for ψ_j. Selberg's conjecture can then be rephrased as the r_j being real and not purely imaginary. (These are the only possiblities since the eigenvalues of Δ are real.) The interesting thing is that

$$(U_m(\cdot,s),U_n(\cdot,w))$$

is equal to (by the unfolding method),

$$\delta_{mn}(4\pi n)^{1-s-\overline{w}}\Gamma(s+\overline{w}-1) + \sum_{c\neq 0}\frac{S(m,n,c)}{|c|^{2s}}\int_0^\infty\int_{-\infty}^\infty \frac{y^{\overline{w}-s}}{(x^2+1)^s}e\left(-\frac{m}{yc^2(x+1)}-ny(x-i)\right)\frac{dxdy}{y^2}$$

It turns out that this is equal to (for $w=\overline{s}+2$):

$$\delta_{mn}(4\pi n)^{-2s-1}\Gamma(2s+1) + 4^{-s-1}\pi^{-1}n^{-2}\frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)}Z(m,n,s) + \sum_{c\neq 0}\frac{S(m,n,c)}{|c|^{2s}}R_{m,n}(s,c)$$

where the latter expression is to be seen as an error term which can be estimated.

In the case under study, we can apply the Selberg spectral decomposition for any function f in $L^2(\Gamma\backslash H)$ as

$$f = \sum_j (f,\psi_j)\psi_j + (part\, involving\, Eisenstein\, series).$$

By Parseval's formula, we have

$$(f,g) = \sum_j (f,\psi_j)(\psi_j,g) + \cdots$$

Thus, putting $f = U_m(\cdot,s)$ and $g = U_n(\cdot,w)$ and choosing $w = \overline{s}+2$ gives the required formula and the theorem of Selberg.

Goldfeld and Sarnak (see [49]) have used this formula to prove a theorem of Kuznetsov:

$$\sum_{c\leq x}\frac{S(m,n,c)}{c} \ll x^{1/6+\epsilon}.$$

Using averaging techniques, which will be discussed in a later lecture, it is possible to show that $\lambda_1 \geq .238$ which is due to Kim and Sarnak [21].

LECTURE 7

Refined Estimates for Fourier Coefficients of Cusp Forms

1 Sieve theory and Kloosterman sums

Last time, we indicated how Kloosterman sums are connected with Selberg's eigenvalue conjecture. This connection has profound implications to questions of classical analytic number theory. Much of the work of Iwaniec reflects this theme.

Kloosterman sums first arose in connection with the circle method. That they are more ubiquitous than first thought is brought out by a fundamental paper of Atkinson on the fourth power moment of the Riemann zeta function. Below, I want to illustrate how Kloosterman sums enter sieve theory and relate the study to the Brun-Titchmarsh theorem. Our discussion will be brief.

The basic set-up of the sieve is as follows. We are given a set $\mathcal{A}$ together with a set of conditions indexed, for notational convenience, by prime numbers $p \in \mathcal{P}$. For each $p \in \mathcal{P}$, we denote by $\mathcal{A}_p$ to be the set of elements of $\mathcal{A}$ which satisfies the conditions indicated by p. The sieve problem is to estimate the size of

$$\mathcal{S}(\mathcal{A},\mathcal{P}) := \mathcal{A}\setminus \cup_{p\in\mathcal{P}} \mathcal{A}_p.$$

If for every squarefree number d composed of primes $p \in \mathcal{P}$, we define

$$A_d := \cap_{p|d} A_p,$$

then, the usual inclusion-exclusion process gives

$$\mathcal{S}(\mathcal{A},\mathcal{P}) = \sum_d \mu(d)|\mathcal{A}_d|$$

when the set $\mathcal{A}$ is finite. This is sometimes referred to as the Sieve of Eratosthenes.

One of the significant applications of sieve theory is to the estimation of the number of primes in a given arithmetic progression. The best result in this direction is given by Montgomery and Vaughan [36] where they show that the number of primes $p \le x$ and $p \equiv a (\bmod c)$ is

$$\le \frac{2x}{\phi(c)\log(x/c)}, \qquad \text{if} \qquad c < x.$$

It is an observation due to Chowla that any improvement in the constant 2 above will imply that there are no "Siegel" zeros. As indicated above, this entails precise estimation of

$$\#\{n \le x : n \equiv a(\bmod c),\ d|n\}.$$

Writing $n = dt$, this means $t \equiv \overline{d}a(\bmod c)$. Now the number of natural numbers $t \le x$ in a fixed residue class $v(\bmod c)$ is easily seen to be

$$\left[\frac{x-v}{c}\right] - \left[\frac{-v}{c}\right].$$

Usually, one writes this as

$$\frac{x}{c} + O(1)$$

and the error term is too crude for many applications. Hooley [14] had the idea that one can Fourier analyse the error term in the following way. Write

$$\psi(x) = x - [x] - 1/2$$

and observe that it has a Fourier series (for x non-integral):

$$\psi(x) = \sum_{h=1}^{\infty} \frac{\sin 2\pi h x}{\pi h}$$

which can be rewritten as a finite sum

$$\sum_{1 \le h \le N} \frac{\sin 2\pi h t}{\pi h} + O\left(\min\left(1, \frac{1}{N||x||}\right)\right).$$

Here $||x||$ denotes the distance from x to the integer nearest to x. Inserting this Fourier series into the sieve method indicated above, one is naturally led to Kloosterman sums.

2 Gauss sums and hyper-Kloosterman sums

The Gauss sum is defined by

$$g(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Notice that

$$\begin{aligned}
\sum_{\chi} \overline{\chi}(b)\chi^2(c) g(\chi)^r &= \sum_{\chi} \overline{\chi}(b)\chi^2(c) \sum_{a_1,\ldots,a_r} \chi(a_1)\cdots\chi(a_r) e^{\frac{2\pi i}{q}(a_1+\cdots+a_r)} \\
&= \sum_{a_1,\ldots,a_r} e^{\frac{2\pi i}{q}(a_1+\cdots+a_r)} \sum_{\chi} \overline{\chi}(b)\chi(c^2 a_1\cdots a_r) \\
&= \phi(q) \sum_{\substack{a_1,\ldots,a_r \\ a_1\ldots a_r \equiv c^{-2} b (\bmod q)}} e^{\frac{2\pi i}{q}(a_1+\cdots+a_r)}
\end{aligned}$$

which is a hyper-Kloosterman sum. Deligne [6], as a consequence of his work on the Weil conjectures has estimated this sum to be $O(q^{(r-1)/2})$. If we normalize our Gauss sums to have absolute value 1, we see that the quantity above is $O(q^{1/2})$. In the method to be discussed in the next section, we will see that hyper-Kloosterman sums enter in a natural way.

3 The Duke-Iwaniec method

Duke and Iwaniec [8] introduced a general method of obtaining estimates for coefficients of Dirichlet series that satisfy appropriate functional equations. We now outline this method.

Let $A = \{a_n\}$ be a sequence of complex numbers. Suppose

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

converges absolutely for $\Re(s) > 1$. For any Dirichlet character χ mod q, let us set

$$A(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}.$$

For technical reasons, we suppose q is prime, so that any non-trivial character mod q is primitive. We assume $A(s, \chi)$ can be analytically continued to an entire function and that it satisfies a functional equation

$$A(1-s, \chi) = \epsilon_\chi \Phi(s) A(s, \overline{\chi})$$

where

$$|\epsilon_\chi| = 1, \quad \Phi(s) \text{ is holomorphic for } \Re(s) > 1.$$

In practice,

$$\Phi(s) = \frac{\gamma(s)}{\gamma(1-s)}$$

where

$$\gamma(s) = (q/\pi)^{ds/2} \prod_{j=1}^{d} \Gamma(s/2 + \mu_j)$$

with

$$\Re(\mu_j) \geq -1/2, \quad \epsilon_\chi = (g(\chi)/\sqrt{q})^d.$$

We will assume that there is a constant $c \geq 1$ such that

$$\Phi(s) \ll (q|s|)^{(2\sigma-1)c}$$

on $\Re(s) = \sigma > 1$ (the implied constant depending on σ). In practice, $2c = d$, as can be seen easily by Stirling's formula when $\gamma(s)$ is given as above. The sign in the functional equation is assumed to be randomly distributed on the unit circle in the following sense. Namely, we suppose that

$$K_q(a) := \sum_{\chi} \overline{\chi}(a) \epsilon_\chi \ll q^{1/2}$$

for all characters χ mod q. (It is possible to weaken this condition and allow for some exceptional characters which are not included in the sum.)

Theorem 7.1 *If these conditons hold for a set of primes q of positive density, then*

$$a_n \ll n^{\frac{2c-1}{2c+1}+\epsilon}$$

for any $\epsilon > 0$.

Proof Let f be a smooth, compactly supported function in $\mathbb{R}^+$. We shall study

$$A_f(q, \ell) = \sum_{n \equiv \ell (\operatorname{mod} q)} a_n f(n)$$

and

$$A_f(\chi) = \sum_n a_n \chi(n) f(n)$$

so that

$$A_f(q, \ell) = \frac{1}{\phi(q)} \sum_{\chi (\operatorname{mod} q)} \overline{\chi}(\ell) A_f(\chi).$$

If we define

$$F(s) = \int_0^\infty f(x)x^{-s}dx$$

then by Mellin inversion formula, we have

$$f(x) = \frac{1}{2\pi i}\int_{(2)} F(s)x^{-s}ds.$$

Thus,

$$A_f(\chi) = \sum_n a_n\chi(n)f(n) = \frac{1}{2\pi i}\int_{(2)} A(s,\chi)F(s)ds.$$

We move the line of integration to $\Re(s) = -1$ and apply the functional equation to get

$$A_f(\chi) = \epsilon_\chi A_g(\overline{\chi})$$

where

$$g(y) = \frac{1}{2\pi i}\int_{(2)} F(s)\Phi(s)y^{-s}ds.$$

Note that g depends on the parity of χ but not on χ otherwise. We will reflect this dependence by writing g_+ and g_- in place of g with self-evident notation. Accordingly, we will split the sum $K_q(a)$ into even and odd characters, getting

$$S_\pm(a) = \frac{1}{2}\left(K_q(a) \pm K_q(-a)\right).$$

We remove the contribution from the trivial character so that

$$A_f(q,\ell) - \frac{1}{\phi(q)}A_f(\chi_0) = \frac{1}{\phi(q)}\sum_{m,\pm} a_m g_\pm(m)S_\pm(\ell m).$$

Thus, this quantity is

$$\ll q^{-1/2}\sum_m |a_m g(m)|.$$

It remains to estimate $g(m)$. To this end, let us assume that the Mellin integral is bounded by

$$|F(s)| \ll |s|^{-r}$$

for some $r > c+1$. Then,

$$g(m) \ll m^{-1-\epsilon}q^{c(1+2\epsilon)}.$$

This gives a final estimate of $O(q^{c-1/2+\epsilon})$.

The above condition on $F(s)$ is easily satisfied with any $r \geq 0$ for the function of type $f(x) = \omega(x/\ell)$ where $\omega(t)$ is a smooth function supported in the interval $[1/2, 2]$ and $q \geq 1$. To see this, we need only integrate by parts the equation defining $F(s)$. We apply the calculation to this test function and obtain

$$\sum_{n\equiv\ell(\bmod q)} a_n\omega(n/\ell) \ll \frac{1}{q}\sum_{n<2\ell}|a_n| + p^{c-1/2+\epsilon}.$$

This holds for primes q in a set of positive density. So we sum it over such primes in an interval of the form $[P, 2P]$. (If q belongs to such an interval, we sometimes denote it by $q \sim P$.) On the left hand side the term $a_\ell\omega(1)$ occurs with high multiplicity. For $n \neq \ell$, there are at most $O(\log \ell)$ prime divisors of $n - \ell$. Thus, summing over $q \sim P$, we get

$$P|a_\ell| \ll \sum_{n<2\ell} |a_n|(\log \ell)^2 + P^{c+1/2+\epsilon}.$$

As the Dirichlet series $A(s)$ converges absolutely for $\Re(s) > 1$, we have

$$\sum_{n\leq x} |a_n| \ll \sum_{n=1}^{\infty} |a_n|(x/n)^{1+\epsilon} \ll x^{1+\epsilon}.$$

Hence,

$$\sum_{n<2\ell} a_n \ll \ell^{1+\epsilon}$$

and we get

$$|a_\ell| \ll \left(\frac{\ell}{P} + P^{c-1/2}\right)\ell^{\epsilon}.$$

Now choose $P = \ell^{2/(2c+1)}$ to deduce the result.

□

LECTURE 8

Twisting and Averaging of L-series

1 Selberg conjectures for GL_n

The general philosophy of the Duke-Iwaniec method is that information about twists of Dirichlet series gives information on the coefficients of the Dirichlet series. This method can also be applied to the study of Γ-factors of twists. Since the functional equation of the L-series attached to a Maass cusp form involves the eigenvalue λ_j in its Γ-factors, we should be able to get some insight into λ_1 by studying twists of the L-series attached to the Maass cusp form. This is the approach of Luo, Rudnick and Sarnak [32] which we discuss in this lecture.

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. If $\mathbb{A}$ is as usual the adele ring of $\mathbb{Q}$, π an irreducible cuspidal automorphic representation of $GL_m(\mathbb{A})$ (with unitary central character), we assume that the archimedean component π_∞ is spherical so that we associate $\mu_{1,\infty}, ..., \mu_{m,\infty}$ and

$$L(s, \pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s - \mu_{j,\infty}).$$

Conjecture 8.1 (Selberg's conjecture for GL_m)

$$\Re(\mu_{j,\infty}) = 0, \quad 1 \le j \le m.$$

Our goal in this lecture will be to prove the following theorem:

Theorem 8.2 *Let π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with π_∞ spherical. Then,*

$$|\Re(\mu_{j,\infty})| \le \frac{1}{2} - \frac{1}{m^2+1}.$$

As a corollary, we get

$$\lambda_1 \ge \frac{21}{100} = .21$$

which follows from the above theorem by taking the Gelbart-Jacquet lift $Sym^2(\pi)$. Indeed, let $\lambda_1 = 1/4 - r^2$. Then, $\mu_{1,\infty} = r, \mu_{2,\infty} = -r$. Since π cannot be monomial (as these have $\lambda \ge 1/4$), it lifts to a cuspidal automorphic representation Π of GL_3 whose Π_∞ is also spherical and is parametrized by

$$\text{diag}(2r, 0, -2r).$$

We apply the theorem to get

$$\Re(2r) \le \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$$

which gives $\Re(r) \le 1/5$. Thus, $\lambda_1 = 1/4 - r^2 \ge 21/100$.

The key to the proof is Rankin-Selberg theory.

$$L(s,\pi_\infty\times\tilde{\pi}_\infty)=\prod_{j,k=1}^{m}\Gamma_{\mathbb{R}}(s-\mu_{j,\infty}-\overline{\mu}_{k,\infty})$$

Let

$$\beta_0=2\max\Re(\mu_{j,\infty}).$$

Then,

$$L(s,\pi_\infty\times\tilde{\pi}_\infty)$$

is holomorphic for $\Re(s)>\beta_0$ and has a pole at $s=\beta_0$. If χ is a primitive even Dirichlet character, then the same is true for the gamma factor of $L(s,(\pi\otimes\chi)_\infty\times\tilde{\pi}_\infty)$. For χ even, primitive of sufficiently large (prime) conductor q, we have

$$\pi\otimes\chi\not\simeq\pi$$

and so

$$L(s,\pi_\infty\times\tilde{\pi}_\infty)L(s,\pi\otimes\chi\otimes\tilde{\pi})$$

is entire. Hence, β_0 is a trivial zero of

$$L(s,\pi\otimes\chi\otimes\tilde{\pi}).$$

That is

$$L(\beta_0,\pi\otimes\chi\otimes\tilde{\pi})=0$$

for all such χ. Now the strategy is to apply the techniques of [40] or [16] to deduce

$$\sum_{q\sim Q}\sum_{\chi\neq\chi_0,\chi\text{ even}}L(\beta,\pi\otimes\chi\otimes\tilde{\pi})\gg\frac{Q^2}{\log Q}$$

for $\Re(\beta)>1-\frac{2}{m^2+1}$ to get a contradiction. To show this, we use the functional equation to approximate

$$L(\beta,\pi\otimes\chi\otimes\tilde{\pi}).$$

Using this approximation, one then averages over the even characters and establishes the result. We will give the details of the method in the next lecture.

2 Ramanujan conjecture for GL_n

If π is a cuspidal automorphic representation of $GL_m(\mathbb{A})$ with local Satake parameters $\alpha_{1,p},\ldots,\alpha_{m,p}$ for p unramified, then we have

Conjecture 8.3 (Ramanujan conjecture for GL_n)

$$|\alpha_{i,p}|=1,\quad 1\le i\le m.$$

It is possible to extend the method above for Selberg's conjecture to treat the Ramanujan conjecture. More precisely, fix a prime p at which π is unramified. The local L-factor

$$L(s,\pi_p\times\tilde{\pi}_p)=\prod_{j,k=1}^{m}(1-\alpha_j(p)\overline{\alpha_k(p)}p^{-s})^{-1}$$

has a pole at β_0 defined via

$$p^{\beta_0}=\max_j|\alpha_j(p)|^2.$$

Hence, the partial L-function

$$L^{(p)}(s,\pi\otimes\tilde{\pi}):=L(s,\pi_p\times\overline{\pi_p})^{-1}L(s,\pi\times\tilde{\pi})$$

has a trivial zero at $s = \beta_0$. The same is true for the twists

$$L^{(p)}(s, (\pi \otimes \chi) \otimes \tilde{\pi})$$

for any χ of conductor q for which $\chi(p) = 1$. This puts the finite and infinite places on the same footing.

Thus, the Ramanujan conjecture would follow from another line of attack. Namely, given π an irreducible cuspidal automorphic representation and β with $\Re(\beta) > 0$, there is an even character χ such that $L(\beta, \pi \otimes \chi) \neq 0$.

3 The method of averaging L-functions

We now give a brief idea of how the method can be used to establish Theorem 8.2. The details will be given in the next lecture. We sketch the idea now.

The Rankin-Selberg theory gives us precise functional equations of the twists. Let $f \in C_c^\infty(0, \infty)$ with

$$\int_0^\infty f(x)dx = 1.$$

Set

$$k(s) = \int_0^\infty f(y)y^s \frac{dy}{y}.$$

Thus, $k(s)$ is entire, rapidly decreasing in vertical strips and $k(0) = 1$. The approximate functional equation gives

$$L(\beta, \chi) = \sum_{n=1}^\infty \frac{b(n)\chi(n)}{n^\beta} F_1(n/Y) + \text{other terms}$$

where

$$F_1(x) = \frac{1}{2\pi i} \int_{(2)} k(s)x^{-s} \frac{ds}{s}.$$

The essential point is we can approximate $L(\beta, \chi)$ with a finite sum of length Q^{m^2}.

Moreover,

$$\sum_{\chi \neq \chi_0, \chi \text{ even}} \chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod q \\ (q-1)/2 - 1 & n \equiv \pm 1 \bmod q \\ -1 & \text{otherwise} \end{cases}$$

Thus,

$$\sum_{q \sim Q} \sum_{\chi \neq \chi_0, \chi \text{ even}} \sum_n \frac{b(n)\chi(n)}{n^\beta} F_1(n/Y) = \sum_{q \sim Q} \frac{q-1}{2} \sum_{n \equiv \pm 1 \pmod q} \frac{b(n)}{n^\beta} F_1(n/Y)$$

$$- \sum_{q \sim Q} \sum_{(n,q)=1, n \not\equiv \pm 1 \bmod q} \frac{b(n)}{n^\beta} F_1(n/Y).$$

The contribution from $n = 1$ gives rise to the dominant term for an appropriate choice of Q. We will give the details in the next lecture.

LECTURE 9

The Kim-Sarnak Theorem

1 Preliminaries

In this lecture, our goal is to establish the best estimates on the Selberg eigenvalue conjecture and the Ramanujan conjecture for GL_n due to Kim and Sarnak [22]. Before we do so, let us examine the averaging idea assuming the Lindelöf hypothesis for automorphic L-functions. This conjecture predicts that

$$L(1/2+it,\pi) = O(f(\pi)^\epsilon(|t|+2)^\epsilon)$$

where $f(\pi)$ denotes the conductor of π.

Given a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

we can write by partial summation

$$f(s) = \sum_{n\le x} \frac{a_n}{n^s} + s\int_x^\infty \frac{S(t)dt}{t^{s+1}}$$

where

$$S(t) = \sum_{n<t} a_n.$$

If χ is a primitive Dirichlet character mod q, suppose that

$$f(s,\chi) = \sum_{n=1}^{\infty} a_n\chi(n)/n^s$$

extends to an entire function and satisfies a "Lindelöf hypothesis" of the form

$$f(1/2+it,\chi) = O(q^\epsilon(|t|+2)^\epsilon)$$

then standard methods of analytic number theory show that

$$S(t,\chi) := \sum_{n\le t} a_n\chi(n) \ll t^{1/2}q^\epsilon.$$

Thus, by what was said above, we find

$$f(\beta,\chi) = \sum_{n\le x} a_n\chi(n)n^{-\beta} + O(q^\epsilon x^{1/2-\beta}).$$

Now, let us consider an averaging

$$\sum_{\chi \text{even}, \chi\ne\chi_0} f(\beta,\chi) = \sum_{n\le x} a_n n^{-\beta}\left(\sum_{\chi \text{even},\chi\ne\chi_0}\chi(n)\right) + O(q^{1+\epsilon}x^{1/2-\beta}).$$

The inner sum is equal to $\phi(q)/2-1$ if $n \equiv \pm 1 \bmod q$ and -1 if $\not\equiv \pm 1 \bmod q$, so that if we choose $x=q$, we get

$$\sum_{\chi \text{even}, \chi \neq \chi_0} f(\beta,\chi) = \frac{\phi(q)-1}{2} + O(q^{1-\beta}) + O(q^{3/2-\beta}) + O(q^{1-\beta+\epsilon}).$$

If $f(\beta,\chi)=0$ for all $\chi \neq \chi_0$, we get a contradiction if $\beta > 1/2$.

Now let π be a cuspidal automorphic representation and let us apply this result to

$$f(s) = \prod_{p<\infty} L(s, \pi_p \times \tilde{\pi}_p).$$

By the method to be described below, we will get for the Ramanujan and Selberg conjectures the following estimates:

$$|\Re(\mu_{j,\infty})| \leq 1/4$$

as well as

$$|\alpha_j(p)| \leq p^{1/4}$$

for the Satake parameters. This holds for general GL_n. If we apply this estimate to the symmetric fourth-power L-function attached to a cuspidal automorphic representation on GL_2, the $1/4$ in the above estimates can be improved to $1/8$. The challenge is to do this calculation without the Lindelöf hypothesis. This is the context of the paper by Luo, Rudnick and Sarnak [32].

2 Rankin-Selberg theory

Let π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$. For π_∞ spherical (or unramified), the gamma factor of $L(s,\pi)$ is

$$L(s,\pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s-\mu_{j,\infty})$$

where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Selberg's conjecture is the assertion that $\Re(\mu_{j,\infty})=0$ for $j=1,...,m$.

If π corresponds to a Maass form of eigenvalue $\lambda = 1/4+r^2$, then $\mu_{1,\infty} = ir$, $\mu_{2,\infty} = -ir$. Selberg's conjecture is then the statement that r is not purely imaginary. In other words, $\Re(\mu_{j,\infty})=0$. The gamma factor of $L(s,\pi\times\overline{\pi})$ is

$$L(s,\pi\times\tilde{\pi}_\infty) = \prod_{j,k=1}^{m} \Gamma_{\mathbb{R}}(s-\mu_{j,\infty}-\mu_{k,\infty}).$$

Let $\beta_0 = 2\max\Re(\mu_{j,\infty})$, then $L(s,\pi_\infty\times\tilde{\pi}_\infty)$ is holomorphic for $\Re(s)>\beta_0$. If χ is a primitive even Dirichlet character, then the same is true for $L(s,(\pi\times\chi)_\infty\times\tilde{\pi}_\infty)$. For χ even, primitive of sufficiently large prime conductor q, we have $\pi\times\chi \not\simeq \pi$ so that

$$L(s,\pi_\infty\times\tilde{\pi}_\infty)L(s,\pi\times\chi\times\tilde{\pi})$$

is entire. Hence, β_0 is a "trivial" zero of $L(s,\pi\times\chi)$ Thus,

$$L(\beta_0,\pi\times\chi\times\overline{\pi}) = 0$$

for all such χ. In this way, the problem becomes the familiar one of proving that certain twists of L-functions do not vanish at a given point. We will prove that

$$\sum_{q\sim Q}\sum_{\chi\neq\chi_0,\,\chi\text{ even}} L(\beta,\pi\times\chi\times\overline{\pi}) \gg \frac{Q^2}{\log Q}$$

for $\Re(\beta) > 1-\frac{2}{m^2+1}$. This is the basic strategy. The same strategy can be applied to improve estimates on the Ramanujan conjecture at the finite primes. Indeed, for p unramified, we have

$$L(s,\pi_p\times\tilde{\pi}_p) = \prod_{j,k=1}^{m}(1-\alpha_j(p)\overline{\alpha_k(p)}p^{-s})^{-1}.$$

Suppose

$$p^{\beta_0} = \max_j |\alpha_j(p)|^2.$$

Then,

$$L(s,\pi_p\times\tilde{\pi}_p)$$

has a pole at $s=\beta_0$. Hence, the partial L-function

$$L^{(p)}(s,\pi\times\tilde{\pi}) = L(s,\pi_p\times\tilde{\pi}_p)^{-1}L(s,\pi\times\pi)$$

has a trivial zero at $s=\beta_0$. The same is true for all twists

$$L^{(p)}(s,\pi\times\chi\times\tilde{\pi})$$

for characters χ with $\chi(p)=1$. By choosing special q's as in [48], one deduces the analogous theorem.

Thus, this argument puts both the finite and infinte versions of the Ramanujan conjectures on the same footing.

3 An application of the Duke-Iwaniec method

We begin by noting that if

$$L(s,\pi\times\tilde{\pi}) = \sum_{n=1}^{\infty} b(n)n^{-s}$$

and

$$L(s,\pi\times\chi\times\tilde{\pi}) = \sum_{n=1}^{\infty} b(n)\chi(n)n^{-s}$$

then the twisted L-function satisfies a functional equation of the form

$$\Lambda(s,\pi\times\chi\times\tilde{\pi}) = \epsilon(s,\pi\times\chi\times\tilde{\pi})\Lambda(1-s,\pi\times\overline{\chi}\times\tilde{\pi})$$

where the global epsilon factor is given by

$$\epsilon(s,\pi\times\chi\times\tilde{\pi}) = \chi(f(\pi\times\tilde{\pi}))\epsilon(s,\pi\times\tilde{\pi})\epsilon(s,\chi)^{m^2}$$

and this can be shown to be equal to

$$\chi(f(\pi\times\tilde{\pi}))\tau(\chi)^{m^2}q^{-m^2 s}\epsilon(s,\pi\times\tilde{\pi})$$

which involves a bit of representation theory (see [32]).

We now apply the argument of Duke and Iwaniec [8]. Let $f\in C_c^{\infty}(0,\infty)$ with

$$\int_0^{\infty} f(x)dx = 1.$$

Set

$$k(s) = \int_0^\infty f(y) y^s \frac{dy}{y}.$$

Thus, $k(s)$ is entire, rapidly decreasing and $k(0) = 1$. For $x > 0$, let

$$F_1(x) = \frac{1}{2\pi i} \int_{(2)} k(s) x^{-s} \frac{ds}{s}$$

and

$$F_2(x) = \frac{1}{2\pi i} \int_{(2)} k(-s) G(-s+\beta) x^{-s} \frac{ds}{s}$$

where

$$G(s) = \frac{L(1-s, \pi_\infty \times \tilde{\pi}_\infty)}{L(s, \pi_\infty \times \tilde{\pi}_\infty)}.$$

Recall that

$$\beta_0 = 2 \max_j \Re(\mu_\infty(j))$$

and we assume $0 < \Re(\beta) < 1$.

Lemma 9.1 1. *$F_1(x)$ and $F_2(x)$ are rapidly decreasing as $x \to \infty$.*
2. *As $x \to 0$,*

$$F_1(x) = 1 + O(x^{-N})$$

for all $N \geq 1$.
3. *As $x \to 0$,*

$$F_2(x) \ll 1 + x^{1-\beta_0-\Re(\beta)-\epsilon}.$$

Proof The asymptotics for $F_1(x)$ follow upon shifting the contour of integration to the right (for $x \to \infty$) and to the left for $x \to 0$). As for $F_2(x)$, we apply Stirling's formula to deduce that $G(s)$ is of moderate growth in vertical strips and so we may shift contours. To get the behaviour as $x \to \infty$, we shift the contour to the right. For the behaviour as $x \to 0$, we shift the contour to the left. If $\Re(\beta) + \beta_0 - 1 < 0$, we pick up a simple pole at $s = 0$ which gives $F_2(x) = O(1)$. Otherwise, we pick up the first pole at $s = \beta + \beta_0 - 1$ and there are none to its right. In this case, we get the bound

$$F_2(x) \ll x^{1-\beta_0-\Re(\beta)}(-\log x)^{d-1}$$

where $d \leq m^2$ is the maximal order of a pole of

$$L(s, \pi_\infty \times \tilde{\pi}_\infty)$$

on the line $\Re(s) = \beta_0$. □

The next step is to derive the "approximate functional equation" in the following form. With F_1 and F_2 defined as above, for $\chi \neq \chi_0 \bmod q$, with q coprime to the conductor of π, and $0 < \Re(\beta) < 1$, we have for $\Pi = \pi \times \tilde{\pi}$,

$$\begin{aligned} L(\beta, \Pi \times \chi) \quad &= \quad \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} F_1(n/Y) \\ &\quad + \tau(\pi \times \tilde{\pi})(q^{m^2} f)^{-\beta} \sum_{n=1}^{\infty} \frac{b(n)\tilde{\chi}(n)}{n^{1-\beta}} \chi(f) \tau(\chi)^{m^2} F_2(nY/fq^{m^2}). \end{aligned}$$

To see this, consider the integral

$$\frac{1}{2\pi i}\int_{(2)} k(s)L(s+\beta,\Pi\times\chi)Y^s\frac{ds}{s} = \sum_{n=1}^{\infty}\frac{b(n)\chi(n)}{n^{\beta}}\left(\frac{1}{2\pi i}\int_{(2)}k(s)(Y/n)^s\frac{ds}{s}\right)$$
$$= \sum_{n=1}^{\infty}\frac{b(n)\chi(n)}{n^{\beta}}F_1(n/Y).$$

By the lemma, this converges absolutely and again by the lemma, we may shift the contour to $\Re(s)=-1$. Thus,

$$\frac{1}{2\pi i}\int_{(2)} k(s)L(s+\beta,\Pi\times\chi)Y^s\frac{ds}{s} = L(\beta,\Pi\times\chi)+\frac{1}{2\pi i}\int_{(-1)} k(s)L(s+\beta,\Pi\times\chi)Y^s\frac{ds}{s}.$$

Applying the functional equation to the second integral, we get

$$\frac{1}{2\pi i}\int_{(-1)} k(s)\tau(\pi\times\tilde{\pi})\chi(f)\tau(\chi)^{m^2}(fq^{m^2})^{-s-\beta}G(s+\beta)L(1-s-\beta,\pi\times\overline{\chi})Y^s\frac{ds}{s}.$$

We now change s to $-s$ and integrate term by term to get

$$\tau(\pi\times\tilde{\pi})\chi(f)\tau(\chi)^{m^2}(fq^{m^2})^{-\beta}\sum_{n=1}^{\infty}\frac{b(n)\overline{\chi}(n)}{n^{1-\beta}}F_2(nY/fq^{m^2}).$$

We sum this over the non-trivial even characters mod q and apply the orthogonality relation noted before, to obtain several sums. The first sum to consider is

$$\sum_{q\sim Q}\sum_{\chi\neq\chi_0,\,\chi\text{ even}}\sum_{n}\frac{b(n)\chi(n)}{n^{\beta}}F_1(n/Y) = \sum_{q\sim Q}\frac{q-1}{2}\sum_{n\equiv\pm1(q)}\frac{b(n)}{n^{\beta}}F_1(n/Y)$$
$$-\sum_{q\sim Q}\sum_{(n,q)=1}\frac{b(n)}{n^{\beta}}F_1(n/Y).$$

We single out the contribution from $n=1$:

$$\sum_{q\sim Q}\frac{q-1}{2}F_1(1/Y)=\sum_{q\sim Q}\frac{q-1}{2}(1+O(Y^{-N}))\sim\frac{cQ^2}{\log Q}$$

for some positive constant c as we will choose Y so that $Q\ll Y\ll Q^{m^2}$. In fact, we will choose

$$Y\asymp Q^{(m^2+1)/2}.$$

The sum over $n\equiv 1 \bmod q$ with $n\neq 1$ gives

$$\sum_{q\sim Q}\sum_{n\equiv1 \bmod q,n\neq1}\frac{b(n)}{n^{\beta}}F_1(n/Y)=\sum_{n}\frac{b(n)}{n^{\Re(\beta)}}F_1(n/Y)\left(\sum_{q\sim Q,q|(n-1),\,n\neq1}\frac{q-1}{2}\right)$$

which is

$$\ll Q\sum_{n}\frac{b(n)n^{\epsilon}}{n^{\Re(\beta)}}|F_1(n/Y)|,$$

where we have used the fact that for $n\neq 1$, the number of representations $n=1+dq=1+d_1q_1$ for fixed n is $O(n^{\epsilon})$ for any $\epsilon>0$. Now use

$$F_1(x)\sim 1$$

as $x\to 0$ to get that this is

$$\ll QY^{1-\Re(\beta)+\epsilon}.$$

Similarly, the same estimate holds for terms $n \equiv -1 \bmod q$. To treat the second terms arising from the approximate functional equation, we use

$$\sum_{\chi \neq \chi_0, \chi \text{ even}} \overline{\chi}(n)\chi(f)\tau(\chi)^{m^2} \ll q^{(m^2+1)/2}$$

by Deligne's bounds for hyperkloosterman sums. Thus, we get

$$\sum_{q \sim Q} (fq^{m^2})^{-\beta} \sum_{\chi \neq \chi_0, \chi \text{ even}} \frac{b(n)\overline{\chi}(n)}{n^{1-\beta}} \chi(f)\tau(\chi)^{m^2} F_2(nY/fq^{m^2}),$$

which by Deligne's bound is

$$\ll \sum_{q \sim Q} (fq^{m^2})^{-\Re(\beta)} \sum_{(n,q)=1} \frac{b(n)}{n^{1-\Re(\beta)}} q^{(m^2+1)/2} F_2(nY/fq^{m^2}).$$

This is easily estimated by partial summation as

$$\ll \sum_{q \sim Q} (fq^{m^2})^{-\Re(\beta)} q^{(m^2+1)/2} \int_1^\infty F_2(Yt/fq^{m^2}) \frac{dt}{t^{1-\Re(\beta)}}$$

upon using the fact that

$$\sum_{n \leq x} b(n) \ll x.$$

Now using the bound for $F_2(x)$ provided by the lemma leads to a final estimate of

$$\ll Q^{1+(m^2+1)/2} Y^{-\Re(\beta)}$$

because F_2 is rapidly decreasing. With our choice of Y, we see that the main term is bigger than the error term if

$$\beta > 1 - \frac{2}{m^2+1}.$$

This leads to:

Theorem 9.2 *Let π be a cuspidal automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with π_∞ spherical. Then,*

$$|\Re(\mu_{j,\infty})| \leq \frac{1}{2} - \frac{1}{m^2+1}.$$

In a similar way, by following the Duke-Iwaniec method [8] one gets the estimate

$$|\log_p |\alpha_{j,p}|| \leq \frac{1}{2} - \frac{1}{m^2+1}.$$

In [22] the method described is actually applied to

$$f(s) = L(s, \pi, \mathrm{Sym}^2),$$

which was shown by Kim [19] to be holomorphic if π is not self-contragredient. The functional equation was established by Shahidi [56]. If χ is a Dirichlet character of conductor q which we take to be prime and large, we have

$$L(s, \pi \times \chi, \mathrm{Sym}^2) = L(s, \pi, \mathrm{Sym}^2 \times \chi^2)$$

so that as long as χ not one of at most two characters mod q, $\pi \times \chi$ is not self-contragredient. It will be noted that the positivity of the $b(n)$ was not used in a vital way and only the weaker estimate

$$\sum_{n \leq x} b(n) \ll x$$

was used. Thus, we may apply the method to $f(s)$ above and deduce as in [22] the following:

Theorem 9.3 *Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$. For π_∞ unramified,*

$$|\Re(\mu_{j,\infty})| \leq \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1},$$

and for $p < \infty$ at which π_p is unramified,

$$|\log_p |\alpha_{j,p}|| \leq \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1}.$$

Applying this to GL_2 over the rational number field gives

$$|\Re(\mu_{j,\infty})| \leq \frac{7}{64}, \quad j = 1, 2$$

when π_∞ is unramified. If $p < \infty$ and π_p is unramified, we have

$$|\log_p |\alpha_{j,p}|| \leq \frac{7}{64}, \quad j = 1, 2.$$

For the Selberg eigenvalue conjecture, this translates as

$$\lambda_1 \geq \frac{975}{4096} = .238...$$

For the general number field, one has the weaker bound of $1/2 - 1/(n^2+1)$ (see [33]).

LECTURE 10

Introduction to Artin L-functions

1 Hecke L-functions

Dirichlet's work on primes in arithmetic progression gave birth to a family of L-functions attached to characters of the group of coprime residue classes mod q. Using the analyticity of these L-functions, and most importantly, their non-vanishing at $s = 1$, Dirichlet deduced the infinitude of primes in a given arithmetic progression $a \pmod q$ with $(a, q) = 1$.

If one wants to generalize Dirichlet's theorem, several natural questions arise. First, the ring of integers O_K of a number field K does not, in general, have the unique factorization property. Thus, we must speak of prime ideals rather than prime elements. Having decided this, the next question is to understand the notion of a residue class. The natural object to take is the ideal class group of a number field and inquire if there are infinitely many prime ideals in a given ideal class. This was the approach taken by Hecke.

Given a number field K, and an ideal $\mathfrak{q}$, we have the notion of the $\mathfrak{q}$-ideal class group defined as follows. We consider the group of fractional ideals of K which are coprime to $\mathfrak{q}$ modulo the principal ideals (α) with $\alpha \equiv 1 \bmod \mathfrak{q}$. Thus, a natural generalization of Dirichlet's theorem is to inquire if there are infinitely many prime ideals in a given $\mathfrak{q}$-ideal class.

In the case $K = \mathbb{Q}$, all ideals are principal and the q-ideal class group is easily seen to be $(\mathbb{Z}/q\mathbb{Z})^*/\pm 1$. Thus, we don't realize $(\mathbb{Z}/q\mathbb{Z})^*$ as a q-ideal class group. Clearly the problem arises with the choice of generator for the principal ideals. To rectify this, we introduce the real embeddings of K in the following way.

We introduce the notion of a generalized ideal $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_\infty$ where $\mathfrak{f}_0$ is an ordinary ideal and $\mathfrak{f}_\infty$ is a collection of real embeddings of K. The $\mathfrak{f}$-ideal class group consists of the group of fractional ideals coprime to $\mathfrak{f}_0$ modulo the subgroup of principal fractional ideals (α) with

$$\alpha \equiv 1 \bmod {}^{\times}\mathfrak{f}$$

which means that $\alpha \equiv 1 \bmod \mathfrak{f}_0$ and $\sigma(\alpha) > 0$ for all $\sigma \in \mathfrak{f}_\infty$. If $K = \mathbb{Q}$, and ∞ denotes the usual embedding of $\mathbb{Q}$ into $\mathbb{R}$, then $q\infty$-ideal class group of $\mathbb{Z}$ retrieves the coprime residue classes mod q. From this perspective, Dirichlet's theorem is to be viewed as a special case of a theorem about the distribution of prime ideals in generalized ideal classes.

The special case when $\mathfrak{f} = \mathfrak{f}_\infty$ includes all the real embeddings, the ideal class group is called the narrow ideal class group and its order is called the narrow class number.

Following Hecke, we may now consider characters of $\mathfrak{f}$-ideal class groups and for each character χ, we set

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

These L-functions are referred to as Hecke L-functions of finite type and χ is called a character of finite order. Hecke showed that these L-functions extend to entire functions and satisfy a suitable functional equation. He also proved that they do not vanish on $\Re(s) = 1$. Thus, following the Tauberian theorem explained in Lecture 1, we deduce that for any generalized $\mathfrak{f}$-ideal class, there are infinitely many prime ideals in that class.

But Hecke considered a more general question in his researches. To any idele class character χ of $\mathbb{A}_K^*/K^*$, he showed how one can associate an L-function, extend it to an entire function and establish a functional equation. These characters are called "grossencharacters" and if we view the idele class group as $GL_1(\mathbb{A}_K)/GL_1(K)$, then Hecke's work is the first level in the Langlands program. For more on Hecke L-functions, see [28].

2 Artin L-functions

Let K/k be finite Galois extension of algebraic number fields. Let $G = \text{Gal}(K/k)$. For each prime ideal $\mathfrak{p}$ of K, let us consider a prime ideal $\wp$ of K dividing $\mathfrak{p}$. We define the decomposition group $D_\wp$ and inertia groups $I_\wp$ by

$$D_\wp = \{\sigma \in G : \wp^\sigma = \wp\}$$

and

$$I_\wp = \{\sigma \in G : \sigma(x) \equiv x^{N(\mathfrak{p})} \bmod \wp \quad \text{for all } x \in O_K\}$$

respectively. Clearly $I_\wp$ is a normal subgroup of $D_\wp$ and one can show that

$$D_\wp/I_\wp \simeq \text{Gal}((O_K/\wp)/(O_k/\mathfrak{p}))$$

as the latter is the Galois group of a finite extension of a finite field. As such, the latter is a cyclic group generated by the Frobenius automorphism

$$\text{Frob}_\mathfrak{p} : x \mapsto x^{N(\mathfrak{p})}.$$

The pull-back of this element to $D_\wp$ which is well-defined up to an element of $I_\wp$ is called the Frobenius element (denoted $\sigma_\wp$) attached to $\wp$. As we vary over the $\wp$ with $\wp|\mathfrak{p}$, the elements $\sigma_\wp$ determine a conjugacy class $\sigma_\mathfrak{p}$ of elements which we call the Artin symbol attached to $\mathfrak{p}$. Of course, this is only well-defined when the inertia group is trivial. In general, it is well-defined modulo inertia.

In the special case that $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{D})$, the Artin symbol turns out to be the Legendre symbol (D/p).

A natural question to ask is the following. Given a conjugacy class C of G, how often do we have $\sigma_\mathfrak{p} \in C$? The Chebotarev density theorem states that

$$\#\{\mathfrak{p},\, N(\mathfrak{p}) \le x \ : \sigma_\mathfrak{p} \in C\} \sim \frac{|C|}{|G|}\frac{x}{\log x}$$

as x tends to infinity. One way to prove this theorem (although historically this was not the case) is to introduce the non-abelian L-series of Artin as follows. Let V be a finite dimensional vector space over $\mathbb{C}$ and let

$$\rho : G \to GL(V)$$

be a representation. The Artin L-function is defined as

$$L(s, \rho; K/k) = \prod_{\mathfrak{p}} \det(1 - \rho(\sigma_{\wp}) N(\mathfrak{p})^{-s} | V^{I_{\wp}})^{-1}.$$

Sometimes, we write $L(s, \chi, K/k)$ for $L(s, \rho, K/k)$ where $\chi = \operatorname{tr} \rho$ is the character of ρ.

Artin's conjecture is the assertion that if ρ is irreducible and unequal to the trivial representation, then $L(s, \rho, K/k)$ extends to an entire function. In the case that ρ is one-dimensional, Artin showed that there is a Hecke character ψ of k of finite order, so that

$$L(s, \rho, K/k) = L(s, \psi).$$

This is usually referred to as Artin's reciprocity law. If K is a quadratic extension of k, then this theorem is precisely the law of quadratic reciprocity for algebraic number fields.

Langlands [30] has enunciated a more general conjecture. Namely, given ρ of degree n, he predicts that there is an automorphic representation $\pi(\rho)$ of $GL_n(\mathbb{A}_k)$ with

$$L(s, \rho, K/k) = L(s, \pi(\rho))$$

and the latter L-function, being a GL_n L-function has been shown by Godement and Jacquet [11] to extend to an entire function. This conjecture of Langlands is referred to as the Langlands reciprocity conjecture or sometimes as the strong Artin conjecture. By the work of Arthur-Clozel [1], we know that it holds for any nilpotent Galois extension K/k.

Before we state what is currently known about this conjecture, it will be useful to review some functorial properties of Artin L-functions. They are:

1. $L(s, 1, K/k) = \zeta_k(s)$;
2. $L(s, \chi_1 + \chi_2, K/k) = L(s, \chi_1, K/k) L(s, \chi_2, K/k)$;
3. if η is a character of a subgroup H of G, then

$$L(s, \eta, K/K^H) = L(s, \operatorname{Ind}_H^G \eta, K/k);$$

4. if M/k is Galois with $M \subseteq K$, and τ is a character of $\operatorname{Gal}(M/k)$ then,

$$L(s, \tau, M/k) = L(s, \tilde{\tau}, K/k).$$

Properties (1), (2) and (4) are easy to verify. Property (3) involves some group theory and algebraic number theory. The details can be found in [28].

Motivated by Artin's conjecture, Brauer [3] was led to prove the following fundamental theorem in group theory. Let G be a finite group and χ any character of G. Then, there exist nilpotent subgroups H_i of G, one-dimensional characters ψ_i of H_i and integers n_i so that

$$\chi = \sum_i n_i \operatorname{Ind}_{H_i}^G \psi_i.$$

An immediate consequence is:

Theorem 10.1 (Brauer, 1947) *The Artin L-function $L(s, \rho, K/k)$ can be written as a quotient of products of Hecke L-functions and consequently, it extends to a meromorphic function.*

Proof We first use Brauer's theorem to write $\chi = \operatorname{tr}\rho$ as a sum of abelian characters induced from nilpotent subgroups. By property (2),

$$L(s, \chi, K/k) = \prod_i L(s, \operatorname{Ind}_{H_i}^G \psi_i, K/k)^{n_i}.$$

By property (3), each of the factors can be written as

$$L(s, \operatorname{Ind}_{H_i}^G \psi_i, K/k) = L(s, \psi_i, K/K^{H_i})^{n_i}.$$

By Artin's reciprocity law, $L(s, \psi_i, K/K^{H_i})$ is a Hecke L-function $L(s, \eta_i)$ (say), which by Hecke's theorem extends to an entire function. This completes the proof of the theorem. □

3 Automorphic induction and Artin's conjecture

Our goal now is to show how a special case of the automorphic induction conjecture in the Langlands program suffices to establish the Langlands reciprocity conjecture and consequently Artin's conjecture.

Conjecture 10.2 (Automorphic induction of Hecke characters) *Let K/k be an arbitrary finite extension of algebraic number fields. If ψ is a Hecke character of K, then there is a cuspidal automorphic representation $\pi(\psi)$ of K such that*

$$L(s, \psi) = L(s, \pi(\psi)).$$

Now we can prove:

Theorem 10.3 *If we have automorphic induction of Hecke characters, then the Langlands reciprocity conjecture follows.*

Proof By the proof of Brauer's theorem, we may write

$$L(s, \chi, K/k) = \prod_i L(s, \psi_i)^{n_i}$$

where ψ_i are Hecke characters of some extension K^{H_i} of k. By the automorphic induction conjecture, we may write

$$L(s, \psi_i) = L(s, \pi_i)$$

for some automorphic representation π_i of k. After regrouping some factors if necessary, we may write

$$L(s, \chi, K/k) = \prod_i L(s, \pi_i)^{e_i}$$

where all of the π_i's are distinct cuspidal automorphic representations. (Here, we are viewing the L-function $L(s, \pi)$ as a product over the finite primes only.) Writing

$$L(s, \pi) = \sum_{\mathfrak{n}} \frac{a_\pi(\mathfrak{n})}{N(\mathfrak{n})^s}$$

and comparing coefficients of Dirichlet series for a prime ideal $\mathfrak{p}$ in the penultimate equality, we get

$$\chi(\sigma_{\mathfrak{p}}) = \sum_i e_i a_{\pi_i}(\mathfrak{p}).$$

Now we compare poles of the "Rankin-Selberg" L-function of both sides. The order of the pole is equal to the multiplicity of the trivial character in $\chi\overline{\chi}$. As

χ is irreducible, the multiplicity is one. On the other hand, the right hand side contributes a pole (by Rankin-Selberg theory) of order

$$\sum_i e_i^2.$$

Thus, all the $e_i = 0$ with one exception e_1 (say) which must be ± 1. If $e_1 = -1$, the Artin L-function would have "trivial poles" at certain negative integers and by Brauer's theorem, we know that all the poles of an Artin L-function lie in the critical strip. □

LECTURE 11

Zeros and Poles of Artin L-functions

1 The Heilbronn character

Heilbronn [13] introduced an important idea in the study of zeros and poles of Artin L-series. We begin by reviewing his fundamental observation. Let K/k be a Galois extension with group G. Fix $s_0 \in \mathbb{C}$. We denote by $\hat{G}$ the set of irreducible characters of G. For each subgroup H of G and $\psi \in \hat{H}$, we let

$$n(H, \psi) = \operatorname{ord}_{s=s_0} L(s, \psi, K/K^H).$$

We define the Heilbronn character as

$$\theta_H(g) = \sum_{\psi \in \hat{H}} n(H, \psi)\psi(g).$$

We begin with the following fundamental proposition whose proof is based on Frobenius reciprocity [54].

Proposition 11.1

$$\theta_G|_H = \theta_H.$$

Proof

$$\begin{aligned}
\theta_G|_H &= \sum_{\chi \in \hat{G}} n(G, \chi)\chi|_H \\
&= \sum_{\psi \in \hat{H}} \left(\sum_{\chi \in \hat{G}} n(G, \chi)(\chi|_H, \psi) \right) \psi \\
&= \sum_{\psi \in \hat{H}} \left(\sum_{\chi \in \hat{G}} n(G, \chi)(\chi, \operatorname{Ind}_H^G \psi) \right) \psi \quad \text{by Frobenius reciprocity}
\end{aligned}$$

The inner sum is

$$\sum_{\chi \in \hat{G}} (\chi, \operatorname{Ind}_H^G \psi) \operatorname{ord}_{s=s_0} L(s, \chi, K/k).$$

Since

$$L(s, \psi, K/K^H) = L(s, \operatorname{Ind}_H^G \psi, K/k)$$

and

$$\operatorname{Ind}_H^G \psi = \sum_{\chi \in \hat{G}} (\chi, \operatorname{Ind}_H^G \psi)\chi$$

we see that

$$L(s, \psi, K/K^H) = \prod_{\chi \in \hat{G}} L(s, \chi, K/k)^{(\chi, \operatorname{Ind}_H^G \psi)}.$$

Thus, the inner sum is equal to $n(H, \psi)$ and the lemma is proved. $\square$

2 The fundamental inequality

Following [9], we consider

$$(\theta_g, \theta_G) = \frac{1}{|G|} \sum_{g \in G} |\theta_G(g)|^2.$$

Since,

$$\theta_G = \sum_{\chi \in \hat{G}} n(G, \chi)\chi$$

we immediately get by the orthogonality relations

$$(\theta_G, \theta_G) = \sum_{\chi \in \hat{G}} n(G, \chi)^2.$$

By by the proposition

$$\theta_G(g) = \theta_{\langle g \rangle}(g).$$

Now, if H is abelian

$$\theta_H(g) = \sum_{\psi \in \hat{H}} n(H, \psi)\psi(g)$$

and by Artin's reciprocity law, $n(H, \psi) \geq 0$. As $|\psi(g)| = 1$ we find

$$|\theta_G(g)| \leq \operatorname{ord}_{s=s_0} \zeta_K(s)$$

because

$$\zeta_K(s) = \prod_{\psi \in \hat{H}} L(s, \psi)$$

in the abelian case. Thus, we have proved:

Theorem 11.2

$$\sum_{\chi \in \hat{G}} n(G, \chi)^2 \leq \left(\operatorname{ord}_{s=s_0} \zeta_K(s)\right)^2.$$

This simple theorem, due to Foote-Murty [9], has some astounding corollaries.

Corollary 11.3 *If $s_0 \neq 1$, and $\zeta_K(s_0) \neq 0$, then every Artin L-function $L(s, \chi, K/k)$ is regular at $s = s_0$.*

Corollary 11.4 *If $\chi \neq 1$, then $L(1, \chi, K/k) \neq 0$.*

Proof If $s_0 = 1$, then $\operatorname{ord}_{s=s_0} \zeta_K(s) = -1$ and we know $n(G, 1) = -1$. Thus, $n(G, \chi) = 0$ for all $\chi \neq 1$. $\square$

Corollary 11.5 *If $\chi \neq 1$, $L(s, \chi, K/k)$ extends to a regular function for $\Re(s) \geq 1$ and non-vanishing there.*

Proof We know that the Dedekind zeta function is analytic for $\Re(s) \geq 1$, $s \neq 1$, and non-vanishing there. Thus, $L(s, \chi, K/k)$ is regular and non-vanishing for $\Re(s) \geq 1$. $\square$

Remark 11.6 The last corollary is important to establish the Chebotarev density theorem using the standard Tauberian argument (see Lecture 1) as well as [28].

Corollary 11.7 (Aramata-Brauer theorem) *$\zeta_K(s)/\zeta_k(s)$ is an entire function.*

Proof We have

$$|n(G,1)| \leq \operatorname{ord}_{s=s_0} \zeta_K(s).$$

But for $s_0 \neq 1$, $|n(G,1)| = n(G,1)$ and for $s_0 = 1$, $\zeta_k(s)$ has a simple pole at $s_0 = 1$. Thus, we deduce that $\zeta_K(s)/\zeta_k(s)$ is entire. □

We have the following important

Conjecture 11.8 (Dedekind) *If $k \subseteq K$, then $\zeta_K(s)/\zeta_k(s)$ is entire.*

The Aramata-Brauer theorem shows that Dedekind's conjecture is true for Galois extensions K/k. It is also known if K is contained in a solvable extension of k.

3 Rankin-Selberg property for Galois representations

The foregoing discussion allows us to deduce the following important property of Galois representations.

Theorem 11.9 *If χ_1 and χ_2 are two irreducible characters of $\mathrm{Gal}(K/k)$, then $L(s, \chi_1\overline{\chi_2}, K/k)$ has a pole at $s = 1$ if and only if $\chi_1 = \chi_2$, in which case the pole is simple.*

Proof By the discussion of the preceding section, the order of the pole of any Artin L-function $L(s, \chi, K/k)$ at $s = 1$ is given by $(\chi, 1)$. In our case, this is $(\chi_1\overline{\chi_2}, 1) = (\chi_1, \chi_2)$ which by the orthogonality relations is equal to 1 or zero according as $\chi_1 = \chi_2$ or not. □

LECTURE 12

The Langlands-Tunnell Theorem

1 Review of some group theory

Let k be an algebraic number field. The Langlands-Tunnell theorem states that if

$$\rho : \mathrm{Gal}(\overline{k}/k) \to GL_2(\mathbb{C})$$

is an irreducible representation such that the image of ρ is solvable, then there is a cuspidal automorphic representation $\pi = \pi(\rho)$ of $GL_2(\mathbb{A}_k)$ such that

$$L(s, \rho) = L(s, \pi).$$

Consequently, both the Artin and Langlands reciprocity conjectures are true in this case. Below, we will sketch two proofs of this theorem and refer the reader to [47] for complete details. The first proof works only when $k = \mathbb{Q}$ and is based on the Deligne-Serre theorem:

Theorem 12.1 (Deligne-Serre) *Let f be a holomorphic newform of weight one with Nebentypus on $\Gamma_0(N)$. Then, there is a Galois representation*

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{C})$$

such that

$$L(s, \rho_f) = L(s, f).$$

First, we recall a classical theorem of Klein (see Lang [29]) that describes the finite solvable subgroups of $PGL_2(\mathbb{C})$. They are either cyclic, dihedral, A_4, S_4 or A_5. In the last three cases, we speak of the tetrahedral, octahedral and icosahedral cases respectively, as these groups can also be realised as the group of symmetries of the tetrahedron, octahedron and icosahedron. As A_5 is not solvable, only the first four cases occur in our situation.

We leave as exercises the following facts. A_4 has a normal 2-Sylow subgroup V isomorphic to the Klein four group. A_4 has no elements of order 6. It has a unique 3-dimensional representation

$$\rho_{tet} = \mathrm{Ind}_V^{A_4} \theta$$

where θ is any non-trivial character of V.

Recall that $GL_2(\mathbb{C})$ acts on the 3-dimensional vector space

$$sl_2(\mathbb{C}) = \{2 \times 2 \,\text{matrices over}\, \mathbb{C}, \text{with trace } 0\}$$

by conjugation

$$g \cdot A \mapsto g^{-1} A g.$$

This gives a representation

$$Ad : GL_2(\mathbb{C}) \to GL_3(C).$$

If $\rho : G \to GL_2(\mathbb{C})$ is any representation, we denote by $Ad(\rho)$ the representation $Ad \circ \rho$ which is a 3-dimensional representation. It is easy to verify that if $\rho(g)$ has eigenvalues α, β then $Ad(\rho)(g)$ has eigenvalues $\alpha/\beta, 1, \beta/\alpha$.

2 Some representation theory

Let E/F be cyclic extension of prime degree ℓ. Recall that we have the base change and automorphic induction maps given as follows:

$$BC_{E/F} : \mathcal{A}(F) \to \mathcal{A}(E)$$

$$AI_E^F : \mathcal{A}(E) \to \mathcal{A}(F)$$

satisfying

$$L(s, BC_{E/F}(\pi)) = \prod_{\chi} L(s, \pi \otimes \chi)$$

with the product over the (abelian) characters of $\mathrm{Gal}(E/F)$ and

$$L(s, AI_E^F(\Pi)_v) = \prod_{w|v} L(s, \Pi_w)$$

for unramified v. We refer the reader to Langlands [31] and Arthur-Clozel [1] for further details and proofs of these maps. Below, we will use the notation $\mathcal{A}(n, E)$ to denote the space of automorphic representations of $GL_n(\mathbb{A}_E)$ and we set

$$\mathcal{A}(E) = \cup_{n \geq 1} \mathcal{A}(n, E)$$

a notation which we have already used above.

The Gelbart-Jacquet lift is a lifting of automorphic forms from GL_2 to GL_3. What they prove [10] is that given $\pi \in \mathcal{A}(F)$ there exists $Ad(\pi)$ which is an automorphic representation of $GL_3(\mathbb{A}_F)$ so that $Ad(A_v(\pi))$ has the same set of eigenvalues as $A_v(Ad(\pi))$. Gelbart and Jacquet [10] also prove that $Ad(\pi)$ is cuspidal if and only if π is not of the form $AI_E^F(\theta)$ with E/F quadratic and θ a Hecke character of E.

We review the notion of Galois invariance of automorphic representations. Let E/F be a finite Galois extension. Let Π be an automorphic representation of $GL_n(\mathbb{A}_E)$. Let η be an element of $\mathrm{Gal}(E/F)$ and $f \in V_\pi$, the representation space corresponding to π. Then, $\mathrm{Gal}(E/F)$ acts on V_π via the action

$$(\eta \cdot f)(g) = f(\eta^{-1} g).$$

This gives the notion of $\eta(\Pi)$. We say that Π is Galois invariant if $\eta(\Pi) = \Pi$ for all $\eta \in \mathrm{Gal}(E/F)$. Now suppose E/F is a finite cyclic extension. By the theory of base change[1], if Π is an automorphic representation of $GL_n(\mathbb{A}_E)$ which is Galois invariant, then it must be the base change lift of some π which is an automorphic representation of $GL_n(\mathbb{A}_F)$.

Now let us consider the A_4-case. Let E^V be the field fixed by V. Consider

$$\rho|_{E^V} : \mathrm{Gal}(E/E^V) \to GL_2(\mathbb{C}).$$

A little reflection shows that $\rho|_{E^V}$ is dihedral and so by the work of Artin-Hecke, we deduce that there is a cuspidal automorphic representation Π of $GL_2(\mathbb{A}_{E^V})$ so that

$$L(s, \Pi) = L(s, \rho, E/E^V).$$

Next, one shows that Π is $\mathrm{Gal}(E^V/F)$-invariant. Thus, there exists a $\pi \in \mathcal{A}(2,F)$ so that

$$\Pi = BC_{E^V/F}(\pi).$$

Therefore

$$\begin{aligned} L(s,\rho,E/E^V) &= L(s,\pi)L(s,\pi\otimes\chi)L(s,\pi\otimes\chi^2) \\ &= L(s,\rho,E/F)L(s,\rho\otimes\chi,E/F)L(s,\rho\otimes\chi^2,E/F). \end{aligned}$$

From this, we see that if v splits completely in E, then $A_v(\pi)$ and $\rho(\mathrm{Frob}_v)$ have the same characteristic polynomial. The difficult case is when v is inert in E^V.

3 An application of the Deligne-Serre theorem

In the case $F = \mathbb{Q}$, it is possible to apply the Deligne-Serre theorem [7] to deduce the theorem immediately. Indeed, choose π so that its central character agrees with $\det\rho$. One checks that π corresponds to a classical holomorphic modular form f of weight one. Hence, by Deligne-Serre, we deduce that there is a Galois representation

$$\rho_f : \mathrm{Gal}(\overline{Q}/\mathbb{Q}) \rightarrow GL_2(\mathbb{C})$$

so that

$$L(s,f) = L(s,\rho_f).$$

We consider the Artin L-functions

$$L(s,\rho\otimes\overline{\rho_f}), L(s,\rho\otimes\overline{\rho_f}\otimes\chi), L(s,\rho\otimes\overline{\rho_f}\otimes\chi^2)$$

and by the Chebotarev density theorem,

$$\sum_{i=0}^{2} \log L(s,\rho\otimes\overline{\rho_f}\otimes\chi^i) \sim 3 \sum_{p \text{ splits in} E} \frac{1}{p^s} \sim \log\left(\frac{1}{s-1}\right)$$

which implies (by our discussion in lecture 11) that ρ is isomorphic to some twist of ρ_f.

4 The general case

In the case of arbitrary ground field, we know by our previous discussion that $Ad(\rho)$ is automorphic over F. That is, $\pi(Ad(\rho))$ exits. What we want to establish is that the characteristic polynomial of $A_v(\pi)$ and $\rho(\mathrm{Frob}_v)$ are the same for all places v. Suppose the eigenvalues of $A_v(\pi)$ are a,b and of $\rho(\mathrm{Frob}_v)$ are α,β. By the determinant condition, we know $ab = \alpha\beta$. From the equality of the base change L-function with the Artin L-function attached to the restriction of ρ to $\mathrm{Gal}(E/E^V)$, we deduce that

$$a = \zeta\alpha, \quad b = \zeta^2\beta$$

for some cube root of unity ζ. From the equality of $Ad(\rho)$ with $\pi(Ad(\rho))$, we deduce that

$$\{a/b, 1, b/a\} = \{\alpha/\beta, 1, \beta/\alpha\}.$$

If $\zeta = 1$, then we are done. If not, $\alpha/\beta = \pm\zeta^2$ by the above. If $\alpha/\beta = \zeta^2$, then the eigenvalues of $A_v(\pi)$ and $\rho(\mathrm{Frob}_v)$ match and we are done. If $\alpha/\beta = -\zeta^2$, then $Ad(\rho(\mathrm{Frob}_v))$ has order 6 which is impossible because A_4 has no elements of order 6. This completes the proof of the tetrahedral case of the Langlands-Tunnell theorem.

For the octahedral case, one needs to use cubic (non-Galois) base change. We refer the reader to [47] for the details.

5 Sarnak's theorem

In this section, we will describe briefly a recent result of Sarnak [50].

As in Sarnak [50], we need to consider first the group-theoretic question of determining the finite subgroups of $GL_2(\mathbb{C})$ which have rational integer determinant and trace. Let

$$GL_2^{(m)}(\mathbb{C}) = \{g \in GL_2(\mathbb{C}) : (\det g)^m = 1\}, \quad m = 1, 2.$$

Up to conjugacy, the finite subgroups of $GL_2(\mathbb{C})$ with rational integer trace and determinant are:

$$U_2 = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \subset GL_2^{(1)}(\mathbb{C})$$

$$V_2 = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \subset GL_2^{(2)}(\mathbb{C});$$

$$\begin{aligned} U_3 &= \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \pm \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \right. \\ &\qquad \left. \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ i & -i \end{pmatrix} \right\} \subset GL_2^{(1)}(\mathbb{C}) \end{aligned}$$

$$\begin{aligned} V_3 &= \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \right. \\ &\qquad \left. \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \right\} \subset GL_2^{(2)}(\mathbb{C}) \end{aligned}$$

$$U_4 = \left\{ \frac{1}{2} \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} | x_j = \pm 1 \right\} \cup U_2.$$

U_2 and V_2 have image V, the Klein four group in $PGL_2(\mathbb{C})$. U_3 and V_3 have image D_3, the dihedral group in $PGL_2(\mathbb{C})$. $U_4 \subset GL_2^{(1)}(\mathbb{C})$ and its image in $PGL_2(\mathbb{C})$ is A_4.

Theorem 12.2 (Sarnak) *Let π be a cuspidal automorphic representation for $GL_2(\mathbb{A}_F)$ whose finite L-series has rational integer coefficients. Then, π corresponds to a Galois representation whose image is solvable. In particular, the Ramanujan conjecture and Selberg's eigenvalue conjecture hold in case π corresponds to a Maass form.*

Proof We follow Sarnak [50]. Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ and χ its central character. For $k = 2, 3, 4$, we will consider when the functorial lifts $\mathrm{Sym}^k(\pi)$ are cuspidal. If $\mathrm{Sym}^2(\pi)$ is not cuspidal, then by [10], there is a quadratic character $\eta \neq 1$ of $\mathbb{A}_F^*/F^*$ such that

$$\pi \otimes \eta \simeq \pi.$$

In such a case, η determines a quadratic extension K of F and π a Hecke character λ of $\mathbb{A}_K^*/K^*$ such that $L(s, \lambda) = L(s, \pi)$ by a theorem of Labesse-Langlands [27]. As π has integer Fourier coefficients, this means that λ must have finite order. If ϵ_λ is the quadratic character of $\mathrm{Gal}(\overline{K}/K)$ corresponding to λ via class field theory, and $\rho = \mathrm{Ind}_K^F \epsilon_\lambda$, then

$$L(s, \rho) = L(s, \lambda) = L(s, \pi),$$

so the theorem is proved in this case. It is not hard to see that

$$L(s, \pi \times \tilde{\pi}) = L(s, \chi) L(s, \mathrm{Sym}^2(\pi)).$$

The Rankin-Selberg L-function has a simple pole at $s = 1$ and so if $\chi \neq 1$, we deduce that $L(s, \mathrm{Sym}^2(\pi))$ has a simple pole at $s = 1$. But then, by Gelbart-Jacquet theorem [10], $\mathrm{Sym}^2(\pi)$ cannot be cuspidal so that we are done by the previous analysis. Thus, we may assume that $\chi = 1$ and that $\mathrm{Sym}^2(\pi)$ is cuspidal. Now if $\mathrm{Sym}^3(\pi)$ is not cuspidal on $GL_4(\mathbb{A}_F)$, then by results of Kim-Shahidi [24] we deduce that π corresponds to a representation ρ of the Weil group W_F of tetrahedral type. But the finite L-series attached to π has integer coefficients and we have $\det \rho = 1$ (as $\chi = 1$). By analysing the possible images in $GL_2(\mathbb{C})$ which have projection into $PGL_2(\mathbb{C})$ equal to A_4, we deduce that π corresponds to a 2-dimensional Galois representation with integer trace and determinant and so must be conjugate to U_4. Thus, we now can assume that $\mathrm{Sym}^2(\pi)$ and $\mathrm{Sym}^3(\pi)$ are cuspidal. If $\mathrm{Sym}^4(\pi)$ is not cuspidal, then by [25], π corresponds to a representation of the Weil group of F of octahedral type. That is, its image in $PGL_2(\mathbb{C})$ is S_4. However, no lift of this group to $GL_2(\mathbb{C})$ can have integer trace and determinant as our earlier discussion shows. Thus, we are left with a π whose central character is trivial, and $\mathrm{Sym}^k(\pi)$ is cuspidal for $k = 2, 3, 4$. We now show that such a π cannot have integer coefficients. Indeed, as in [25], the Rankin-Selberg L-functions $L(s, \mathrm{Sym}^j(\pi) \times \mathrm{Sym}^k(\pi))$ for $2 \leq j, k \leq 4$. The analytic properties including their non-vanishing on $\Re(s) = 1$ are known from [57]. From this, we deduce that each of the L-functions $L(s, \mathrm{Sym}^k(\pi))$ is analytic and non-vanishing on $\Re(s) \geq 1$ for $1 \leq k \leq 8$. By standard Tauberian theorems, we deduce that for any polynomial $T(x)$ of degree at most 8, we have

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \leq N} T(\lambda_\pi(p)) \log p \to \int_{-2}^{2} T(x) d\mu(x), \qquad (*)$$

where

$$d\mu(x) = \frac{1}{\pi} \sqrt{1 - x^2/4}\, dx$$

is the Sato-Tate measure. Now consider

$$T(x) = x^2 (x-1)^2 (x+1)^2 (4 - x^2).$$

Observe that $T(m) \leq 0$ for $m \in \mathbb{Z}$, while $T(x) \geq 0$ for $x \in [-2, 2]$. From the first inequality, we see that if $\lambda_\pi(p) \in \mathbb{Z}$ for all p, then, the left hand side of (*) is less than or equal to zero. Whereas, from the second inequality, we see that the right hand side is positive. This is a contradiction. □

Recently, Booker [2] has proved that if the Artin L-series $L(s, \rho)$ attached to a 2-dimensional Galois representation of icosahedral type has its inverse Mellin transform not a modular form, then $L(s, \rho)$ has infinitely many poles. In particular, Artin's holomorphy conjecture for this representation implies that it is modular. We refer the reader to [2] for details.

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