SOME REMARKS RELATED TO MAEDA'S CONJECTURE

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ABSTRACT. In this article we deal with the problem of counting the number of pairs of normalized eigenforms (f, g) of weight k and level N such that $a_p(f) = a_p(g)$ where $a_p(f)$ denotes the p-th Fourier coefficient of f. Here p is a fixed prime.

1. INTRODUCTION

Let $S_k(N)$ denote the space of cusp forms of weight k and level N on the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$. Let $f(z) = \sum_{n \geq 1} a_n q^n \in S_k(N)$ be a normalized Hecke eigenform, i.e., f is an *eigenfunction* for all the Hecke operators T_p's and U_p's and $a_1 = 1$. The famous Ramanujan-Petersson conjecture (proved by Deligne [2]) says that the p-th Fourier coefficient of f is of the form

$$(1.1) \qquad a_p(f) = 2p^{(k-1)/2} \cos \theta_p(f), \quad p \nmid N,$$

for some real *angle* $\theta_p(f) \in [0, \pi]$.

In this paper we shall discuss the following problem: *for a fixed prime p, count the pairs of normalized eigenforms (f, g) of weight k and level N such that $a_p(f) = a_p(g)$.*

More precisely, we prove the following:

Theorem. *For a fixed prime p, the number of pairs (f, g) of normalized eigenforms in $S_k(N)$ such that $\theta_p(f) = \theta_p(g)$ is bounded by*

$$O\left(\frac{(\dim S_k(N))^2 (\log p)}{(\log kN)} \right),$$

where the implied constant is absolute and independent of p.

In other words, our result gives a small saving over the trivial bound provided $\log p \ll \log kN$.

Our interest in this problem is partly motivated by a famous conjecture of Maeda [3] which predicts for $N = 1$ that the polynomial

$$(1.2) \qquad \prod_f (X - a_p(f)),$$

where the product is over all normalized Hecke eigenforms, is irreducible over the rational number field. In fact, Maeda conjectures that the Galois group of this

polynomial is the full symmetric group $\mathfrak{S}_d$ where d is the dimension of the space $S_k(1)$. In other words, Maeda predicts that for level 1, the number of pairs in our theorem is exactly dim $S_k(1)$. For a fixed higher level, Tsaknias [5] has conjectured that the above polynomial is a product of a bounded number of irreducible polynomials viewed as a function of k. Though there is some computational data to support these conjectures, they seem to be far out of reach of our present knowledge and techniques. Thus, it seems appropriate to investigate these questions through methods currently known. This is partial motivation for our work.

In the course of our proof, a certain exponential sum arises. Based on general heuristics about such exponential sums (more precisely the "philosophy" of *square root* cancellation) we give a heuristic argument to support Maeda-Tsaknias's prediction [5] regarding the bounded number of Galois orbits. Before we proceed to prove the theorem, we recall some preliminary results which will play an important role in proving the theorem.

2. Approximation of characteristic functions with Selberg polynomials

Let $I = [a, b]$ be an interval contained in $[-1/2, 1/2]$ and χ_I the characteristic function of the interval I. From the works of Selberg, Beurling and Vaaler (see [7]), there is a trigonometric polynomial S_M of degree at most M such that the following hold:

(a)

$$\chi_I(x) \leq S_M(x), \tag{2.1}$$

(b)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S_M(x)dx = b - a + \frac{1}{M+1}. \tag{2.2}$$

Let $e(t)$ denote $e^{2\pi i t}$. If we write the Fourier series for $S_M(x)$ as

$$S_M(x) = \sum_{|n| \leq M} \widehat{S}_M(n)e(nx), \tag{2.3}$$

then

(c)

$$|\widehat{S}_M(n)| \leq \frac{1}{M+1} + \min\left(b - a, \frac{1}{\pi|n|}\right). \tag{2.4}$$

3. A preliminary estimate

From now onwards we shall denote by χ_I the characteristic function of the interval $I = [-\delta, \delta] \subseteq [-1/2, 1/2]$, with δ to be chosen later.

From (2.1) it follows that

$$\chi_I\left(\theta_p(f) - \theta_p(g)\right) \leq S_M\left(\theta_p(f) - \theta_p(g)\right). \tag{3.1}$$

By the Fourier series expansion of $S_M(x)$, (3.1) can be written as

$$\chi_I\left(\theta_p(f) - \theta_p(g)\right) \leq \sum_{-M}^{M} \widehat{S}_M(n)e\left(n\left(\theta_p(f) - \theta_p(g)\right)\right). \tag{3.2}$$

Thus the number of normalized Hecke eigenforms $f, g \in S_k(N)$ such that $\theta_p(f) = \theta_p(g)$ is

$$\leq \sum_{f \neq g} \chi_I(\theta_p(f) - \theta_p(g)) + \dim S_k(N) = \sum_{f,g} \chi_I(\theta_p(f) - \theta_p(g)).$$

For reasons that will become apparent later, it is convenient to write this as

$$\leq \frac{1}{2} \sum_{f,g} \chi_I(\pm \theta_p(f) \mp \theta_p(g))$$

since χ_I is an even function. By (3.2), this is

$$\leq \frac{1}{2} \sum_{-M}^{M} \widehat{S}_M(n) \sum_{f,g} e\left(n\left(\pm\theta_p(f) \mp \theta_p(g)\right)\right).$$

This is bounded by

(3.3)
$$\leq \frac{1}{2} \sum_{|n| \leq M} |\widehat{S}_M(n)| \left| \sum_f e(\pm n\theta_p(f)) \right|^2.$$

Recall the estimate (2.4):

$$|\widehat{S}_M(n)| \leq \frac{1}{M+1} + \min\left(b - a, \frac{1}{\pi|n|}\right).$$

Using this, the expression (3.3) for $n = 0$ and for $n \neq 0$ (respectively) is

(3.4)
$$
\begin{aligned}
\leq \quad & \left(2\delta + \tfrac{1}{M+1}\right) (\dim S_k(N))^2 \\
+ \quad & \sum_{1 \leq |n| \leq M} \left\{ \tfrac{1}{M+1} + \min\left(2\delta, \tfrac{1}{\pi|n|}\right) \right\} \left| \sum_f e(\pm n\theta_p(f)) \right|^2.
\end{aligned}
$$

The crucial exponential sum that was elucidated in the beginning is

$$\sum_f e(\pm n\theta_p(f)) = \sum_f 2\cos n\theta_p(f).$$

It is this sum that appears in the Eichler-Selberg trace formula and is thus amenable to estimation. In the next sections, we examine ways to estimate this sum.

4. A HEURISTIC ARGUMENT

We shall first give a heuristic argument to show that the estimate in the theorem is bounded by $O((\dim S_k(N))$. In his 1997 paper [6], Serre proved that the θ_p's are *equidistributed* as f varies with respect to a *p-Sato-Tate measure*. This was made effective with the error term by Ram Murty and Kaneenika Sinha in [4] using the Eichler-Selberg trace formula. More precisely, they gave precise estimates for

$$\sum_f 2\cos n\theta_p(f) - c_n \dim S_k(N),$$

where $c_0 = 1$, and $c_n = p^{-n/2} - p^{-(n-2)/2}$ for n even and zero if n is odd. It may be reasonable to expect *square root* cancellation for the error term, that is,

$$\left| \sum_f e(\pm n\theta_p(f)) - c_n \dim S_k(N) \right| = O\left((\dim S_k(N))^{1/2}\right).$$

Using the elementary inequality $2ab \leq a^2 + b^2$, we have $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, so that

$$\left| \sum_f e(\pm n\theta_p(f)) \right|^2 \ll |c_n|^2 (\dim S_k(N))^2 + \left| \sum_f e(\pm n\theta_p(f)) - c_n \dim S_k(N) \right|^2.$$

With our assumption of square root cancellation, and taking $\delta = 1/M$, equation (3.4) is

$$(4.1) \qquad \ll \frac{(\dim S_k(N))^2}{M + 1} + \sum_{1 \leq |n| \leq M} \left\{ \frac{1}{M + 1} + \min\left(2\delta, \frac{1}{\pi|n|} \right) \right\} \dim S_k(N)$$

$$\leq \frac{(\dim S_k(N))^2}{M + 1} + \frac{2M}{M + 1} \dim S_k(N),$$

by virtue of the convergence of

$$\sum_n c_n^2.$$

Thus, by taking $M = \dim S_k(N)$, we obtain a final estimate of

$$(4.2) \qquad\qquad\qquad O\left(\dim S_k(N) \right).$$

This estimate is consistent with the conjectures of Maeda and Tsaknias. Indeed, in the case of level 1, Maeda predicts that the polynomial (1.2) will be irreducible and so it should have **no** repeated roots. Thus the number of pairs (f, g) such that $\theta_p(f) = \theta_p(g)$ is equal to $\dim S_k(1)$. In the higher level case, Tsaknias predicts the number of Galois orbits will be finite and so the polynomial (1.2) will be a product of a **finite** number of irreducible polynomials with rational coefficients. So in this case also, we expect the number to be bounded by a constant multiple of $\dim S_k(N)$.

5. Proof of the main theorem

What can be proved unconditionally? Recall that we want to find a non-trivial bound for the exponential sum $| \sum_f e(\pm n\theta_p(f)) |^2$ appearing earlier. This is achieved by using the following result of Ram Murty and Kaneenika Sinha (see [4], Theorem 18) which we state as

Lemma 1. *Define $f(N)$ as $\sum_{c|N} \phi(\gcd(c, N/c))$, and denote by $\nu(N)$ the number of distinct prime divisors of N. Let*

$$\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p} \right).$$

Let $c_0 = 1$ and for $m \geq 1$, let

$$c_m = \begin{cases} p^{-m/2} - p^{-(m-2)/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Then,
(5.1)
$$\left| \sum_f 2 \cos m\theta_p(f) - c_m \dim S_k(N) \right| \leq 4p^m 2^{\nu(N)} \sup_{f^2 < 4p^m} \psi(f) + 2f(N) + \delta_m(k),$$

where $\delta_m(k) = 0$ unless $k = 2$ in which case it is equal to $2p^{m/2}$.

As mentioned in (11) of [4], the bound in the previous lemma can be replaced by $p^{3m/2}2^{\nu(N)}\log p^m + \sqrt{N}d(N)$. We have proved above that the quantity in question is given by (3.4). We choose $\delta = 1/M$. Then our quantity is

$$\ll \frac{(\dim S_k(N))^2}{M} + \sup_{1\le|m|\le M}\left|\sum_f e(\pm m\theta_p(f))\right|^2$$

which is

$$\ll \frac{(\dim S_k(N))^2}{M}\left(1 + \sum_{m\ge 1}\frac{1}{p^m}\right) + p^{3M}M^2(\log p)^2 4^{\nu(N)} + Nd^2(N).$$

We need to make an optimal choice of M. As the referee suggests, this is best done by using the Lambert W-function. We refer the reader to [1] (see especially Appendix A) for a friendly introduction to this function. Recall that this function is defined by

$$W(x)e^{W(x)} = x.$$

Our choice of M will be so that the first two terms in the above estimate are comparable. That is, we seek M so that

$$(\dim S_k(N))^2 = p^{3M}M^3(\log p)^2.$$

Since $\dim S_k(N)$ lies somewhere between kN and $kN\log\log N$, our M should (essentially) satisfy

$$(kN)^2(\log p) = p^{3M}M^3(\log p)^3.$$

Thus,

$$M\log p = W\left((kN)^{2/3}(\log p)^{1/3}\right).$$

The Lambert function satisfies

$$W(x) = \log x - \log\log x + o(1).$$

As x tends to infinity, we deduce that M can be taken as the nearest integer to

$$\frac{\frac{2}{3}\log kN + \frac{1}{3}\log p}{\log p}$$

which gives us a final estimate of

$$\ll \frac{(\dim S_k(N))^2 \log p}{\log kN}.$$

This completes the proof.

6. Concluding remarks

There are other predictions of the conjectures of Maeda and Tsaknias. For instance, both conjectures predict that for a fixed N, the number of normalized Hecke eigenforms with integer coefficients will be at most 1 (in the level 1 case) and $O_N(1)$ in the higher level case. In the latter case, the constant is expected to be independent of k. Such questions were investigated in [4] where similar methods and estimates were derived.

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