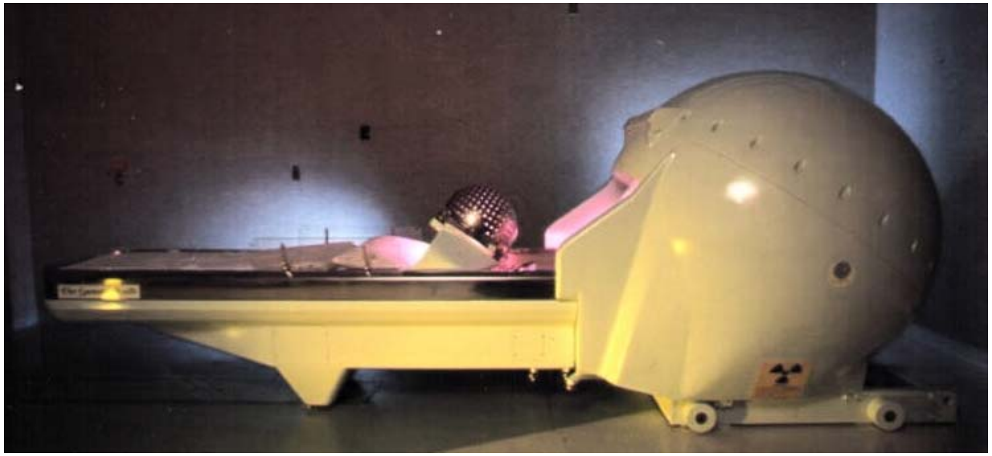
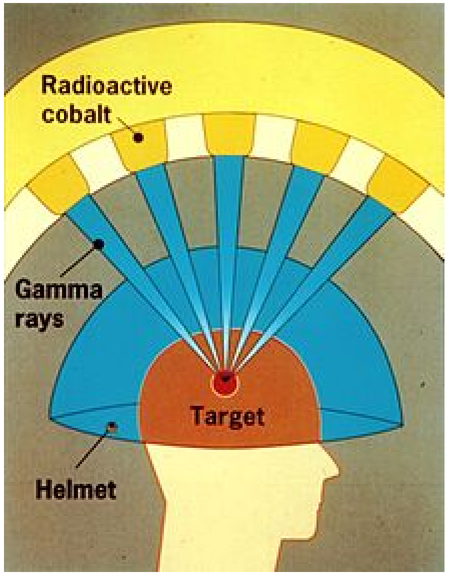
Skeletons and gamma ray radiosurgery

The Mathematics of Shapes



What is gamma-knife surgery?

- It is a non-invasive medical procedure used to treat tumors, usually in the brain.
- This is called radiosurgery since it uses radiation to perform the surgery.
- 201 Cobalt gamma ray beams are arrayed in a hemisphere and aimed through a collimator to a common focal point.
- The patient's head is positioned so that the tumor is the focal point.

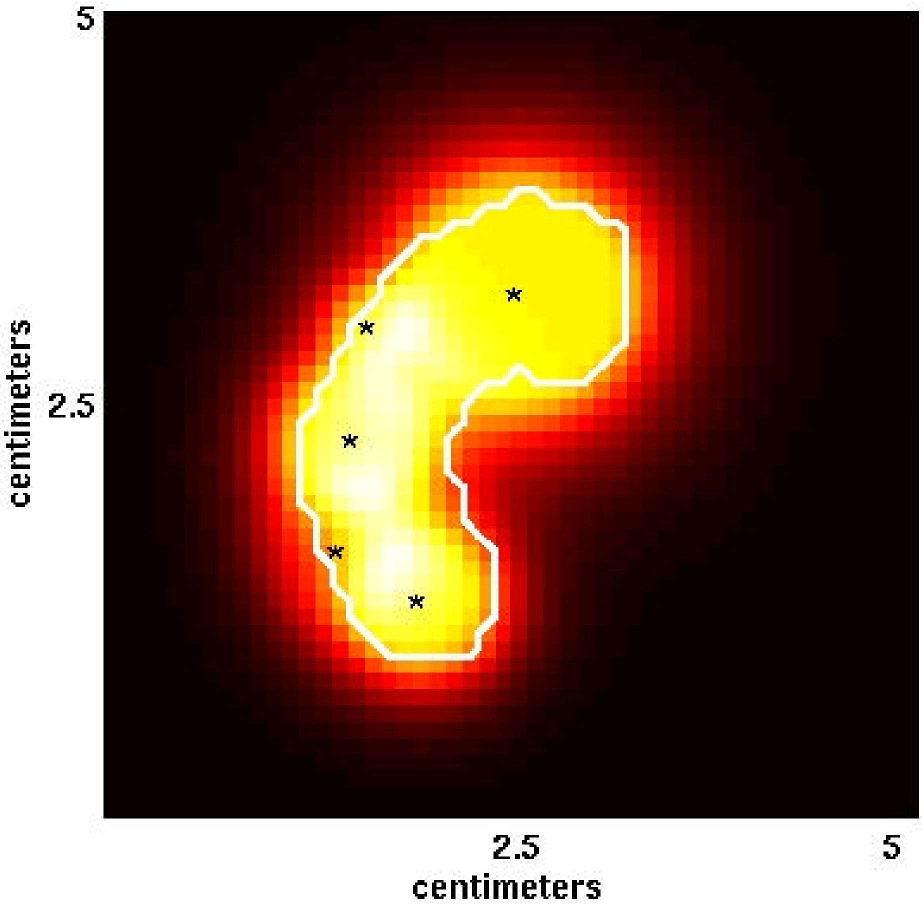


The minimax problem

- Since the tumor maybe of irregular shape and spread over a region, the idea is to minimize the number of radiation treatments and maximize the portion of the area to be treated.
- When the beams are focused with the help of a helmet, they produce focal regions of various sizes.
- Each size of dose requires a different helmet and so the helmet needs to be changed when the dose radius needs to be changed.
- Since each helmet weighs 500 pounds, it is important to minimize the number of helmet changes.

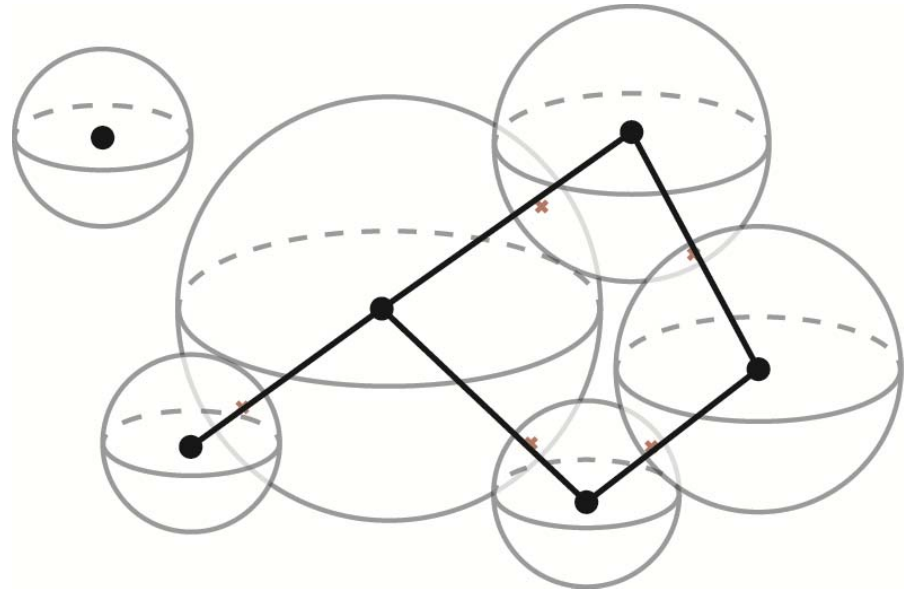
The mathematics of shapes

- Here is the target area on which the radiation is to be applied.
- Since the helmets have varying degrees of focal regions, several helmets have to be used.



Sphere packing problem

- Since we have spheres of different sizes and not all of the affected region can be targeted, the problem can be formulated mathematically as follows:



It is easy to see that this problem is somewhat related to the problem of stacking spheres. We wish to fill (as much as possible) a region $R \subset \mathbb{R}^3$ with spheres in such a way that the proportion of volume not covered is less than some threshold of tolerance ϵ. If we use balls (or solid spheres) $B(X_i, r_i) \subset R$, $i = 1, \ldots, N$, with centers X_i and radii r_i, then the irradiated zone is $P_N(R) = \cup_{i=1}^{N} B(X_i, r_i)$. Letting $V(S)$ represent the volume of a region S, we wish to find balls such that

$$\frac{V(R) - V(P_N(R))}{V(R)} \leq \epsilon. \tag{4.1}$$

The skeleton of a region

- Let |X-Y| denote the Euclidean distance between two points in the plane or in space.

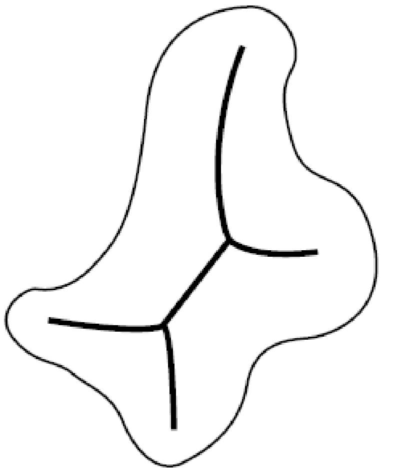


Fig. 4.1. The skeleton of a region.

Thus, if two points X and $Y \in \mathbb{R}^2$ have coordinates (x_1, y_1) and (x_2, y_2) respectively, then the distance between them is

$$|X - Y| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Definition 4.2 *Let R be a region of $\mathbb{R}^2$ (or $\mathbb{R}^3$) and let ∂R be its boundary. The* skeleton *of R, denoted by $\Sigma(R)$, is the following set of points:*

$$\Sigma(R) = \left\{ X^* \in R \,\middle|\, \begin{array}{c} \exists X_1, X_2 \in \partial R \text{ such that } X_1 \neq X_2 \text{ and} \\ |X^* - X_1| = |X^* - X_2| = \min_{Y \in \partial R} |X^* - Y| \end{array} \right\}.$$

Two dimensional skeletons

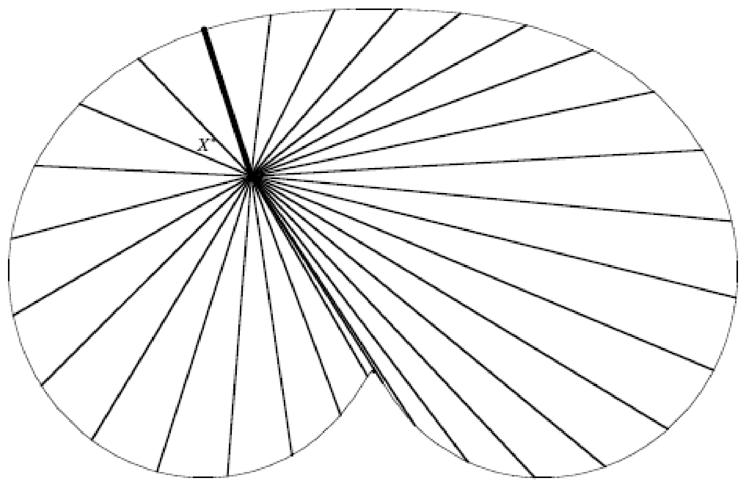
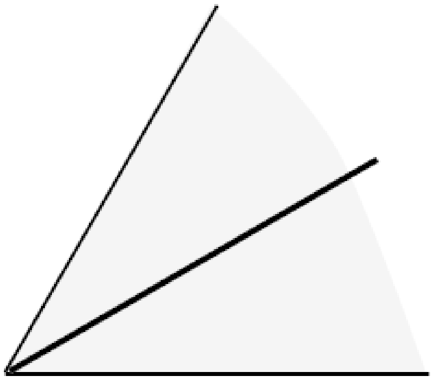


Fig. 4.2. Looking for the shortest distance between a point X^* and the boundary ∂R.

- We denote the skeleton of a region R by **Σ**(R).

Proposition 4.3 *Let $X^* \in R$ and $d = \min_{Y \in \partial R} |X^* - Y|$. Then d is the maximum radius such that $B(X^*, d)$ lies completely within R, i.e., $d = \max\{c > 0 : B(X^*, c) \subset R\}$. The point X^* is in the skeleton $\Sigma(R)$ if and only if $S(X^*, d) \cap \partial R$ contains at least two points.*

Simple skeletons



(a) Skeleton of an angular region.

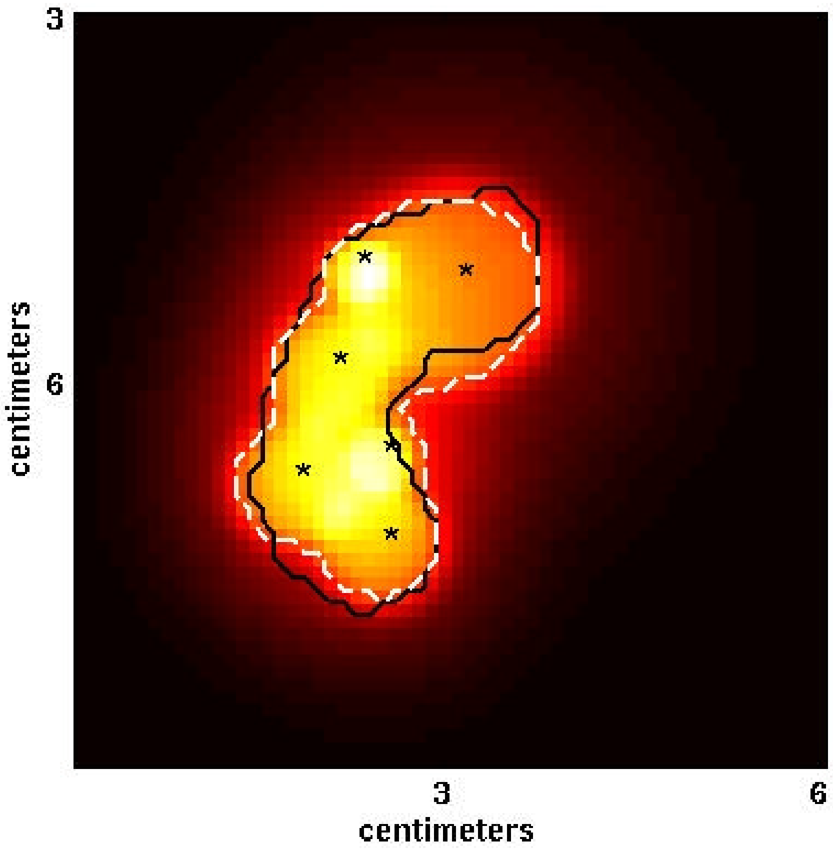
(b) Skeleton of a strip region.

- Given a region in R^2 we want to determine its skeleton since the centers of the focal regions will be situated along the skeleton.

(a) The skeleton

Skeletons in R^3

- The gamma rays will be focused on selected points along the skeleton of the region.



Three dimensional skeletons

- Our earlier definition of a skeleton applies in higher dimensions as well, and in particular to R^3. However, here we can distinguish two portions of the skeleton.

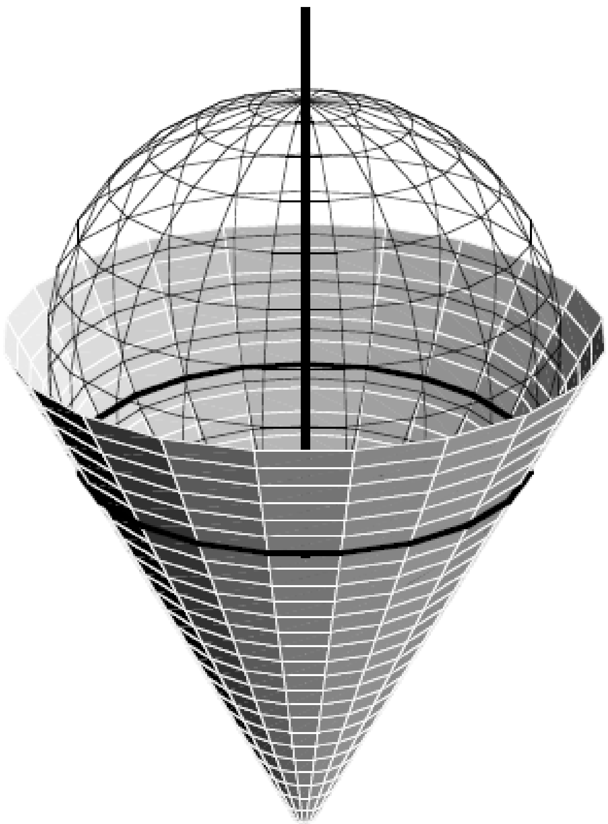
Definition 4.10 *Let R be a region of space and ∂R its boundary. The* linear portion *of the skeleton is defined as*

$$\Sigma_1(R) = \{X^* \in R \mid \exists\, X_1, X_2, X_3 \in \partial R \text{ such that } X_1 \neq X_2 \neq X_3 \neq X_1 \text{ and such that } |X^* - X_1| = |X^* - X_2| = |X^* - X_3| = \min_{X \in \partial R} |X^* - X|\}.$$

The surface portion *of the skeleton of R is*

$$\Sigma_2(R) = \Sigma(R) \setminus \Sigma_1(R).$$

Some simple examples

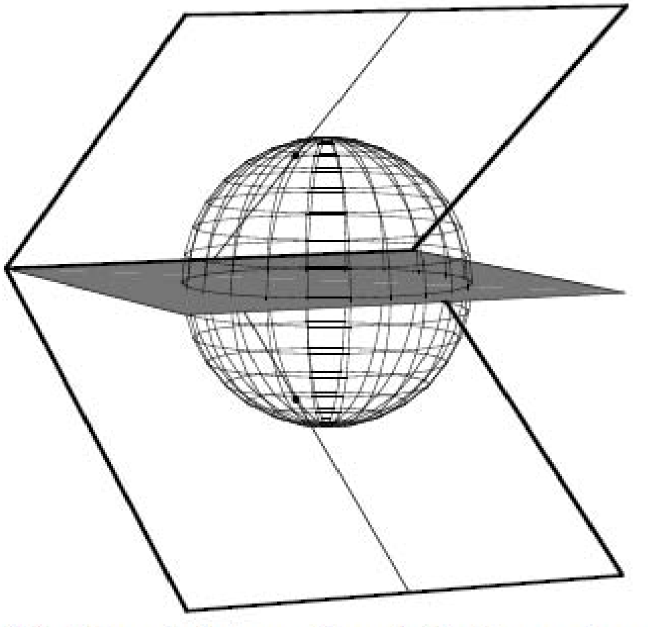


(a) The skeleton of a solid circular cone is given by its central axis

- While the region is the solid filled cone, only the boundary is shown as well as one maximal ball and its circle of tangency.

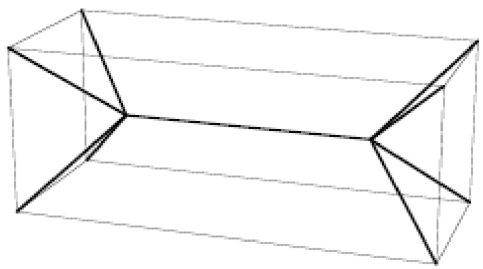
Skeleton of a wedge

- An infinite wedge consists of all points between two half-planes emanating from a common axis. A maximal sphere is shown with its points of tangency.

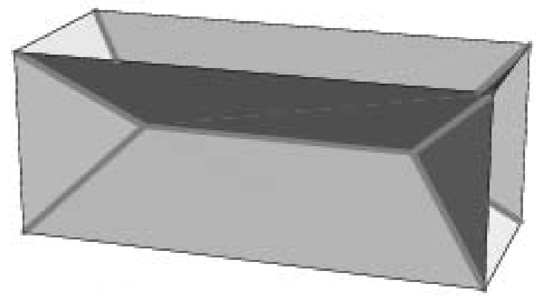


(b) The skeleton of an infinite wedge is given by the bisecting half-plane

Skeleton of a parallelepiped

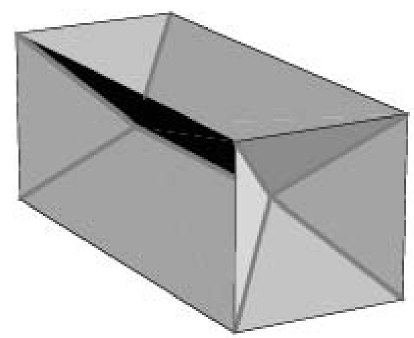


(a) The linear part of the skeleton



(b) The entire skeleton

- These examples are simple since the region is simple to describe. In general, the problem of finding the skeleton of a general region is based on computer algorithms.



(c) A second view of the skeleton

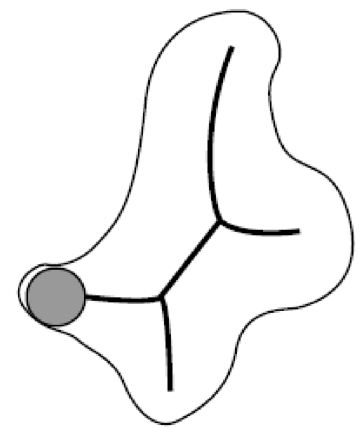
The optimal surgery algorithm

The underlying idea. Suppose that an optimal solution for a region R is given by

$$\cup_{i=1}^{N} B(X_i^*, r_i).$$

Then if $I \subset \{1, \ldots, N\}$, we must have that $\cup_{i \notin I} B(X_i^*, r_i)$ is an optimal solution for $R \setminus \cup_{i \in I} B(X_i^*, r_i)$

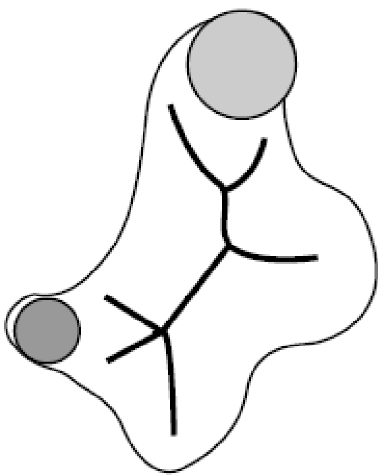
- Any dose in an optimal solution must be centered along the skeleton. If we have four sizes of doses, a<b<c<d (say), then the initial does should be at an extreme point of the skeleton.



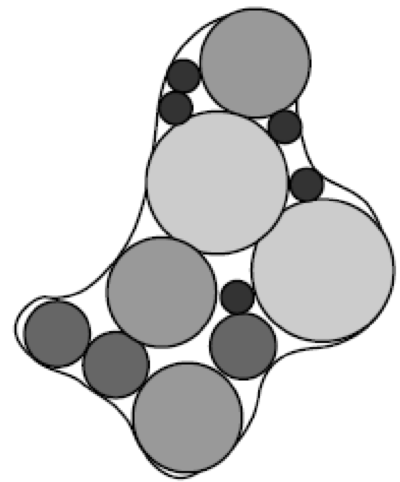
(a) A first dose of radius 4 mm

The iterative procedure

- After the first dose, the region has changed and we need to re-calculate the skeleton.



(b) The skeleton of the remaining region after two doses of radii 4 and 7 mm



(c) The entire region irradiated with doses of radii 2, 4, 7 and 9 mm