Math 402/802: Assignment 1 (due: October 1, 2015)
Students in Math 402: Do any eight questions.
Students in Math 802: Do all questions.

---

1. If $0 \leq k < \lfloor n/2 \rfloor$, show that
$$\binom{n}{k} \leq \binom{n}{k+1}.$$

2. Prove that
$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$
for $n \geq 1$.

3. Show that
$$\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$$
for each $n \geq k \geq \ell \geq 0$.

4. Show that for $k \geq 1$,
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

5. Show that for $n \geq 1$,
$$\binom{n+k+1}{k+1} = \sum_{i=0}^{n} \binom{k+i}{i}.$$

6. For $0 \leq k \leq n$, show that
$$\binom{n}{k} k = n \binom{n-1}{k-1}.$$
Deduce that
$$\sum_{A \subset [n]} |A| = n 2^{n-1}.$$

7. Let F_n denote the n-th Fibonacci number. Show that
$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}.$$

8. The Lucas sequence is defined recursively as follows: $L_1 = 1, L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$. Show that the power series
$$\sum_{n=1}^{\infty} L_n x^n$$
is a rational function and using partial fractions, find an explicit formula for the n-th Lucas number.

9. The multiple zeta values $\zeta(k_1, ..., k_r)$ are defined as

$$\zeta(k_1, ..., k_r) = \sum_{1 \le n_1 < n_2 \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

with $k_1, ..., k_{r-1} \ge 1$ and $k_r \ge 2$. The *weight* of the multiple zeta value $\zeta(k_1, ..., k_r)$ is defined as the sum $k_1 + \cdots + k_r$ and its *depth* as r. A recent remarkable theorem is that **all** multiple zeta values of weight n are $\mathbb{Q}$-linear combinations of

$$\{\zeta(a_1, ..., a_r) : \text{where} \quad a_i = 2 \text{ or } 3, \text{ and } a_1 + \cdots + a_r = n\}.$$

Using this theorem, show that the dimension of the $\mathbb{Q}$-vector space V_n spanned by multiple zeta values of weight n is bounded by d_n where d_n satisfies the recurrence relation $d_n = d_{n-2} + d_{n-3}$, with $d_1 = 0$, $d_2 = 1$. [It is conjectured that the dimension of V_n is exactly d_n but this has not yet been proved.]

10. With d_n as in the previous exercise, find an explicit formula for d_n. Deduce that there are positive constants a, b with $b > 1$ such that d_n is asymptotic to ab^n as n tends to infinity.