Math 402/802: Assignment 3 (due: November 26, 2015)
Students in Math 402: Do any eight questions.
Students in Math 802: Do all questions.

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1. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two locally finite posets. Show that

$$\mu((x_1, y_1), (x_2, y_2)) = \mu(x_1, x_2)\mu(y_1, y_2).$$

2. Find a formula for the number of sequences of length n using the symbols A, B, C, D which have the symbols A, B, C appearing at least once.

3. Recall that the **Bruhat** ordering in S_n is defined as follows. A permutation τ is covered by σ if σ differs from τ by an inversion. That is, σ covers τ if and only if $\sigma(k) = \tau(k)$ apart from two indices i, j where for $i < j$, we have $\sigma(i) > \sigma(j)$. Draw the Hasse diagram for S_3 with the Bruhat order and determine completely the Möbius function of this poset.

4. Let G be a group acting on a set X and H a group acting on a set Y. Assume that X and Y are disjoint and let $U = X \cup Y$. For $g \in G, h \in H$, define

$$(g, h) \cdot x := g \cdot x, \quad \text{if } x \in X$$

and

$$(g, h) \cdot y := h \cdot y, \quad \text{if } y \in Y.$$

Show that this defines an action of $G \times H$ on U.

5. Let G be a finite group acting on a finite set X. We say G acts transitively if given any two elements $x, y \in X$, there is an element $g \in G$ such that $gx = y$. Show that if G acts transitively on X, then

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| = 1.$$

6. For $\sigma \in S_n$, we say σ has cycle type $(c_1, c_2, ..., c_n)$ if σ has precisely c_i cycles of length i in its unique decomposition as a product of disjoint cycles. Show that the number of permutations of type $(c_1, c_2, ..., c_n)$ is

$$\frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!}.$$

7. Let S_3 be the group of permutations of the set $\{1, 2, 3\}$. Let $X = \{1, 2, 3\}$ and define an action of S_3 on X by $\sigma \cdot i = \sigma(i)$ for $i = 1, 2, 3$ and $\sigma \in S_3$. Calculate the cycle index polynomial $P_{S_3}(x_1, x_2, x_3)$.

8. Let G and H be finite groups acting on finite sets X and Y respectively. Assume that X and Y are disjoint. By question 4 above, we can define an action of $G \times H$ on $X \cup Y$. If P_G and P_H indicate the cycle index polynomials of G acting on X and H acting on Y respectively, show that the cycle index polynomial of $G \times H$ acting on $X \cup Y$ is $P_G P_H$.

9. Let p be a prime number. Show that the number of $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ with determinant not divisible by p is given by

$$(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}).$$

10. Let S_n act on the set $X = \{1, 2, ..., n\}$ in the usual way. Let P_{S_n} be the cycle index polynomial. Prove that P_{S_n} is the coefficent of z^n in the power series expansion of

$$\exp\left(zx_1 + z^2x_2/2 + z^3x_3/3 + \cdots\right).$$