1. Let \((P_1, \leq_1)\) and \((P_2, \leq_2)\) be two locally finite posets. Show that 
\[ \mu((x_1, y_1), (x_2, y_2)) = \mu(x_1, x_2) \mu(y_1, y_2). \]

2. Find a formula for the number of sequences of length \(n\) using the symbols \(A, B, C, D\) which have the symbols \(A, B, C\) appearing at least once.

3. Recall that the **Bruhat** ordering in \(S_n\) is defined as follows. A permutation \(\tau\) is covered by \(\sigma\) if \(\sigma\) differs from \(\tau\) by an inversion. That is, \(\sigma\) covers \(\tau\) if and only if \(\sigma(k) = \tau(k)\) apart from two indices \(i, j\) where for \(i < j\), we have \(\sigma(i) > \sigma(j)\). Draw the Hasse diagram for \(S_3\) with the Bruhat order and determine completely the Möbius function of this poset.

4. Let \(G\) be a group acting on a set \(X\) and \(H\) a group acting on a set \(Y\). Assume that \(X\) and \(Y\) are disjoint and let \(U = X \cup Y\). For \(g \in G, h \in H\), define 
\[ (g, h) \cdot x := g \cdot x, \quad \text{if} \ x \in X \]
and
\[ (g, h) \cdot y := h \cdot y, \quad \text{if} \ y \in Y. \]
Show that this defines an action of \(G \times H\) on \(U\).

5. Let \(G\) be a finite group acting on a finite set \(X\). We say \(G\) acts transitively if given any two elements \(x, y \in X\), there is an element \(g \in G\) such that \(gx = y\). Show that if \(G\) acts transitively on \(X\), then
\[ \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| = 1. \]

6. For \(\sigma \in S_n\), we say \(\sigma\) has cycle type \((c_1, c_2, \ldots, c_n)\) if \(\sigma\) has precisely \(c_i\) cycles of length \(i\) in its unique decomposition as a product of disjoint cycles. Show that the number of permutations of type \((c_1, c_2, \ldots, c_n)\) is
\[ \frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!}. \]
7. Let $S_3$ be the group of permutations of the set $\{1, 2, 3\}$. Let $X = \{1, 2, 3\}$ and define an action of $S_3$ on $X$ by $\sigma \cdot i = \sigma(i)$ for $i = 1, 2, 3$ and $\sigma \in S_3$. Calculate the cycle index polynomial $P_{S_3}(x_1, x_2, x_3)$.

8. Let $G$ and $H$ be finite groups acting on finite sets $X$ and $Y$ respectively. Assume that $X$ and $Y$ are disjoint. By question 4 above, we can define an action of $G \times H$ on $X \cup Y$. If $P_G$ and $P_H$ indicate the cycle index polynomials of $G$ acting on $X$ and $H$ acting on $Y$ respectively, show that the cycle index polynomial of $G \times H$ acting on $X \cup Y$ is $P_G P_H$.

9. Let $p$ be a prime number. Show that the number of $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ with determinant not divisible by $p$ is given by

\[(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}).\]

10. Let $S_n$ act on the set $X = \{1, 2, \ldots, n\}$ in the usual way. Let $P_{S_n}$ be the cycle index polynomial. Prove that $P_{S_n}$ is the coefficient of $z^n$ in the power series expansion of

\[\exp \left( z x_1 + z^2 x_2/2 + z^3 x_3/3 + \cdots \right).\]