MATH 401/801: Assignment 3

Due: October 27, 2014

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1. Prove or disprove: The polynomial

$$\lambda^4 - \lambda^2 - 2$$

is the characteristic polynomial of the adjacency matrix of a bipartite graph.

2. Calculate the characteristic polynomial of the cycle graph C_4 on 4 vertices. More generally, calculate the characteristic polynomial of the cycle graph C_n when n is even.

3. Let $d_1, d_2, ..., d_n$ be positive integers. Show that there exists a tree on n vertices with vertex degrees $d_1, d_2, ..., d_n$ if and only if

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

4. Let $T(n; d_1, ..., d_n)$ be the number of labelled trees with n vertices and degree sequence $d_1, ..., d_n$ with $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that

$$T(n; d_1, ..., d_n) = \sum_{j=1}^{n-1} T(n-1; d_1, ..., d_j - 1, ..., d_{n-1}).$$

Deduce that

$$T(n; d_1, ..., d_n) = \frac{(n-2)!}{(d_1 - 1)! \cdots (d_n - 1)!}.$$

5. Using the binomial theorem, prove by induction on r the multinomial theorem which states that

$$(x_1 + \cdots + x_r)^n = \sum_{0 \leq i_1, ..., i_r \leq n} \binom{n}{i_1, ..., i_r} x_1^{i_1} \cdots x_r^{i_r},$$

where

$$\binom{n}{i_1, ..., i_r} := \frac{n!}{i_1! \cdots i_r!}.$$

Deduce from the previous exercise Cayley's theorem, namely, that the number of trees on n labelled vertices is n^{n-2}.

6. (a) Let X be a connected graph on n vertices. Show that X has exactly one cycle if and only if X has n edges.
   (b) Prove that a connected graph with n vertices and e edges contains at least $e - n + 1$ cycles.

7. If X is a graph and e is an edge, we say that e is a **bridge** if $X - e$ has more connected components than X. Prove that in any tree, every edge is a bridge. Deduce that if X is a tree on n labelled vertices, then each element of $\{X - e : e \in E(X)\}$ is a forest of two trees.

8. Let T_n be the number of trees on n labelled vertices. Using the previous exercise (or otherwise), prove that

$$2(n-1)T_n = \sum_{i=1}^{n-1} \binom{n}{i} T_i T_{n-i} i(n-i).$$

Deduce that

$$\sum_{i=1}^{n-1} \binom{n}{i} i^{i-1}(n-i)^{n-i-1} = 2(n-1)n^{n-2}.$$

9. Let $G(r, s; m)$ be the number of connected bipartite graphs with partite sets of size r and s having m edges, and let $F(r, s; m)$ be the number of such graphs not containing any vertices of degree 1. Prove that

$$F(r, s; m) = \sum_{i,j} \binom{r}{i} \binom{s}{j} (-1)^{i+j} G(r-i, s-j; m-i-j)(s-j)^i (r-i)^j.$$

10. Let $T(r, s)$ be the number of spanning trees in the bipartite graph $K_{r,s}$. Prove that

$$0 = \sum_{i,j} \binom{r}{i} \binom{s}{j} (-1)^{i+j} T(r-i, s-j)(s-j)^i (r-i)^j$$

and deduce $T(r, s) = r^{s-1} s^{r-1}$.