1. Prove or disprove: The polynomial
\[ \lambda^4 - \lambda^2 - 2 \]
is the characteristic polynomial of the adjacency matrix of a bipartite graph.

2. Calculate the characteristic polynomial of the cycle graph \( C_4 \) on 4 vertices. More generally, calculate the characteristic polynomial of the cycle graph \( C_n \) when \( n \) is even.

3. Let \( d_1, d_2, \ldots, d_n \) be positive integers. Show that there exists a tree on \( n \) vertices with vertex degrees \( d_1, d_2, \ldots, d_n \) if and only if
\[ \sum_{i=1}^{n} d_i = 2n - 2. \]

4. Let \( T(n; d_1, \ldots, d_n) \) be the number of labelled trees with \( n \) vertices and degree sequence \( d_1, \ldots, d_n \) with \( d_1 \geq d_2 \geq \cdots \geq d_n \). Prove that
\[ T(n; d_1, \ldots, d_n) = \sum_{j=1}^{n-1} T(n-1; d_1, \ldots, d_j-1, \ldots, d_{n-1}). \]
Deduce that
\[ T(n; d_1, \ldots, d_n) = \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}. \]

5. Using the binomial theorem, prove by induction on \( r \) the multinomial theorem which states that
\[ (x_1 + \cdots + x_r)^n = \sum_{0 \leq i_1, \ldots, i_r \leq n} \binom{n}{i_1, \ldots, i_r} x_1^{i_1} \cdots x_r^{i_r}, \]
where
\[ \binom{n}{i_1, \ldots, i_r} := \frac{n!}{i_1! \cdots i_r!}. \]
Deduce from the previous exercise Cayley’s theorem, namely, that the number of trees on \( n \) labelled vertices is \( n^{n-2} \).

6. (a) Let \( X \) be a connected graph on \( n \) vertices. Show that \( X \) has exactly one cycle if and only if \( X \) has \( n \) edges.
(b) Prove that a connected graph with \( n \) vertices and \( e \) edges contains at least \( e - n + 1 \) cycles.
7. If $X$ is a graph and $e$ is an edge, we say that $e$ is a **bridge** if $X - e$ has more connected components than $X$. Prove that in any tree, every edge is a bridge. Deduce that if $X$ is a tree on $n$ labelled vertices, then each element of $\{X - e : e \in E(X)\}$ is a forest of two trees.

8. Let $T_n$ be the number of trees on $n$ labelled vertices. Using the previous exercise (or otherwise), prove that

$$2(n - 1)T_n = \sum_{i=1}^{n-1} \binom{n}{i} T_{n-i}(n-i).$$

Deduce that

$$\sum_{i=1}^{n-1} \binom{n}{i} i^{i-1}(n-i)^{n-i-1} = 2(n-1)n^{n-2}.$$ 

9. Let $G(r, s; m)$ be the number of connected bipartite graphs with partite sets of size $r$ and $s$ having $m$ edges, and let $F(r, s; m)$ be the number of such graphs not containing any vertices of degree 1. Prove that

$$F(r, s; m) = \sum_{i,j} \binom{r}{i} \binom{s}{j} (-1)^{i+j} G(r-i, s-j; m-i-j)(s-j)^i(r-i)^j.$$ 

10. Let $T(r, s)$ be the number of spanning trees in the bipartite graph $K_{r,s}$. Prove that

$$0 = \sum_{i,j} \binom{r}{i} \binom{s}{j} (-1)^{i+j} T(r-i, s-j)(s-j)^i(r-i)^j$$

and deduce $T(r, s) = r^{s-1}s^{r-1}$. 