Math 382: Assignment 1 (due: September 28, 2017)

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1. Show that for all natural numbers $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Show that for all natural numbers $n \geq 1$,

$$1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! = (n+1)! - 1.$$

3. Show that for all natural numbers $n \geq 1$,

$$\int_0^\infty x^n e^{-x} dx = n!$$

4. The Fibonacci sequence is defined as follows. $F_1 = 1$, $F_2 = 1$ and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. Show that

$$F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}.$$

5. Show that for all natural numbers $N \geq 1$,

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}.$$

6. Let p_n denote the n-th prime. Show that for $n \geq 1$,

$$p_n < 2^{2^n}.$$

7. For each natural number n, define

$$I_n = \int_0^{\pi/2} \sin^n \theta d\theta.$$

By integrating by parts, deduce that for $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

Deduce that

$$I_{2n} = \binom{2n}{n} \frac{\pi}{2^{2n+1}} \quad \text{and} \quad I_{2n+1} = \frac{2^{2n} n!^2}{(2n+1)!}.$$

8. If $m = p_1^{a_1} \cdots p_k^{a_k}$ and $n = p_1^{b_1} \cdots p_k^{b_k}$ are the respective unique factorizations, show that
$$\gcd(m,n) = p_1^{\min\{a_1,b_1\}} \cdots p_k^{\min\{a_k,b_k\}},$$
and
$$\operatorname{lcm}(m,n) = p_1^{\max\{a_1,b_1\}} \cdots p_k^{\max\{a_k,b_k\}}.$$
9. Show that for any natural number $n > 1$, the number
$$S := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
is not a natural number. [Hint: consider k such that $n/2 < 2^k \leq n$ and let d be the lcm of all the numbers 1, 2, ..., n except for 2^k and analyse dS.]
10. Let $d = \gcd(m,n)$. Show that
$$\gcd(a^m - 1, a^n - 1) = a^d - 1,$$
for any natural number $a > 1$.