Math 382: Assignment 2 (due: October 17, 2017)

---

1. Prove that
$$2222^{5555} + 5555^{2222}$$
is divisible by 7.
2. Let p be a prime number greater than 3. Show that the numerator of
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{(p-1)^2}$$
(when expressed in lowest terms) is divisible by p
3. Show that a natural number is divisible by 9 if and only if the sum of its digits is divisible by 9.
4. If p is a prime number, show that $\sqrt{p}$ is irrational.
5. For any natural number m, show that
$$\sum_{\substack{j=1 \\ (j,m)=1}}^{m} j = \frac{1}{2} m\phi(m).$$
[Hint: first show that if j is coprime to m, then so is $m - j$.]
6. For $m \geq 4$, show that the numerator of
$$\sum_{\substack{j=1 \\ (j,m)=1}}^{m} \frac{1}{j}$$
(when expressed in lowest terms) is divisible by m.
7. Find the last three digits of 3^{2017}.
8. Show that the number of $j \leq n$ with $(j, n) = d$ is precisely $\phi(n/d)$. Deduce that
$$\sum_{d|n} \phi(d) = n.$$
9. Let p be a prime number greater than 3. Show that the numerator of
$$S := \sum_{j=1}^{p-1} \frac{1}{j}$$
is divisible by p^2. [Hint: Note that $2S = \sum_{j=1}^{p-1} \left(\frac{1}{j} + \frac{1}{p-j} \right)$. and apply question 2.]
10. Prove that if n divides $2^n - 1$, then $n = 1$. [Hint: Suppose not. Let $n_0 > 1$ be the smallest such number and observe that $2^{\phi(n_0)} \equiv 1 \pmod{n_0}$.]