MATH 498/812: Assignment 1

Due: 4 October 2012

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1. Show that if x, y, z are integers such that $x^2 + y^2 + z^2 \equiv 0 \pmod 4$, then x, y, z are all even.

2. Show that any number of the form $4^a(8m + 7)$ cannot be written as a sum of three squares.

3. A number is called **squarefree** if it is not divisible by the square of a prime number. Let D be squarefree. Show that if natural numbers m and n can be written in the form $a^2 + Db^2$, then so can their product.

4. Let $m > 1$. prove that the Schnirelman density of the set of natural numbers $\equiv 1 \pmod m$ is equal to $1/m$.

5. (a) Show that Schnirelman's theorem is false if we replace Schnirelman density by natural density.

   (b) Let $A \subseteq \mathbb{N}$ with Schnirelman density δ. Then, $A(n) \geq \delta n$. Show that this inequality is false if δ is replaced by natural density.

6. Let q be a natural number such that all numbers of the form qn^2 can be written as a bounded number of k-th powers. Show that every natural number can be written as a sum of a bounded number of k-th powers.

7. Let $A \subseteq \mathbb{N}$ be a subset with Schnirelman density δ with $0 < \delta < 1$. Show that every natural number can be written as a sum of at most

$$2\left(1 + \left[-\frac{\log 2}{\log(1-\delta)}\right]\right)$$

   elements from A.

8. (a) Show that

$$\sum_p \frac{1}{p^2} < \frac{1}{2},$$

   where the sum is over all prime numbers p.

(b) Show that the set of squarefree numbers has Schnirelman density $> 1/2$. Deduce that every natural number can be written as a sum of at most two squarefree numbers.

9. Let $r_{g,k}(m)$ be the number of solutions of

$$x_1^k + \cdots + x_g^k = m, \qquad 0 \le x_i \le m^{1/k}.$$

Show that

$$\sum_{m \le n} r_{g,k}(m) \le 2^g n^{g/k}.$$

10. Show that there is a constant $c(g,k) > 0$ such that

$$r_{g,k}(n) \ge c(g,k) n^{g/k-1},$$

for infinitely many n. This shows that Linnik's theorem is essentially best possible.