MATH 498/812: Assignment 2

Due: 25 October 2012

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1. Prove that a natural number n can be written as the difference of two square numbers if and only if $n \not\equiv 2 \pmod 4$.

2. Let $v(k)$ be the smallest value of g such that every natural number n can be written as
$$n = \pm x_1^k \pm x_2^k \pm \cdots \pm x_g^k.$$
Prove that $v(2) = 3$.

3. Prove that
$$\sum_{n \le x} \log n = x \log x - x + O(\log x).$$

4. Prove that there is a constant B such that
$$\sum_{n \le x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + B + O\left(\frac{\log x}{x}\right).$$

5. Define γ as
$$1 - \int_1^\infty \frac{f(t)dt}{t^2},$$
where $f(t) = t - [t]$. Prove that
$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O(1/x).$$

6. Let P, Q denote real numbers with $P > 1$ and $Q \ge 2P$. Show that the intervals
$$\left\{\alpha : \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ}\right\}, \qquad q \le P, \quad 1 \le a \le q, \quad (a,q) = 1,$$
do not overlap.

7. Prove that the number of solutions of the equation
$$x_1 + x_2 + \cdots + x_g = n,$$
where the x_i are non-negative integers, is given by the binomial coefficient
$$\binom{n+g-1}{n}.$$

8. Prove that the number of solutions of

$$x_1 + x_2 + \cdots + x_g \leq n,$$

where the x_i are non-negative integers is given by the binomial coefficient

$$\binom{n+g}{n}.$$

9. Let $G(k)$ denote the smallest value of g such that every sufficiently large number can be written as a sum of g k-th powers. Show that $G(k) \geq k+1$. [Hint: Using the previous exercise, show that there are infinitely many natural numbers that cannot be written as a sum of k k-th powers.]

10. Prove that $G(4) \geq 15$. [Hint: First show that any fourth power is $\equiv 0$ or 1 (mod 16).]