1. Suppose $f \in L^1(\mathbb{R})$ and $f(t) > 0$ for all $t \in \mathbb{R}$. Show that $|\hat{f}(y)| < \hat{f}(0)$ for every $y \neq 0$.

2. Show that
$$\lim_{t \to 0} \int_0^\infty e^{-tx} \sin x \frac{dx}{x} = \lim_{A \to \infty} \int_0^A \sin x \frac{dx}{x} = \frac{\pi}{2}.$$ 

3. Compute the Fourier transform of the function $f \in L^1(\mathbb{R})$ given by $f(x) = 1$ if $x \in [-1,1]$ and zero otherwise. Show that $\hat{f} \in L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$.

4. Prove that
$$\int_{-\infty}^{\infty} e^{itx} dt = \frac{\pi}{1 + t^2}.$$

5. Find the Fourier transform of
$$f(x) = \frac{x}{(1 + x^2)^2}.$$

6. Solve for $f$:
$$\int_{-\infty}^{\infty} f(x-t)e^{-|t|} dt = \frac{4}{3}e^{-|x|} - \frac{2}{3}e^{-2|x|}.$$

7. Compute the Fourier transform of $g(x) = 1 - |x|$ for $|x| \leq 1$ and zero otherwise. Using this, apply the Poisson summation formula to deduce
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

8. In a Hilbert space, show that the maps $x \mapsto (x, y)$ for fixed $y$ and the map $x \mapsto ||x||$ are continuous maps.

9. Suppose $f$ is a continuous function of $\mathbb{R}^1$ with period 1. Prove that
$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t) dt$$
for every irrational real number $\alpha$. [Hint: do it first for $f(t) = e^{2\pi i kt}$, with $k = 0, \pm 1, \pm 2, ...$]

10. For functions $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform
$$\hat{f}(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i(x,t)} dx,$$
where $(x,t)$ denotes the usual inner product and $dx$ is the Lebesgue measure on $\mathbb{R}^n$. If $f(x) = e^{-|x|^2/2}$, show that $\hat{f} = f$. (Here $x = (x_1, ..., x_n)$ and $|x|^2 = x_1^2 + \cdots + x_n^2$.)