1. Let $w \in \mathbb{C}$ be a fixed complex number with $|w| < 1$. Let

$$f(z) = \frac{z - w}{1 - wz}.$$ 

Show that $f$ is regular in $|z| \leq 1$ and calculate $f(w)$ and $f'(w)$.

2. Let $z_1, z_2, \ldots, z_n$ be distinct complex numbers. Let $C$ be the circle around $z_1$ such that $C$ and its interior do not contain $z_j$ for $j > 1$. Let

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n).$$

Evaluate

$$\int_C \frac{dz}{f(z)}.$$

3. Show that equality can occur in the isoperimetric inequality if and only if the simple closed curve is a circle.

4. Prove the following discrete version of integration by parts: For any two sequences of numbers $a_n, b_n$,

$$S_N := \sum_{n=1}^{N} a_n b_n = a_N B_N - \sum_{n=1}^{N-1} B_n (a_{n+1} - a_n),$$

where

$$B_n = \sum_{j=1}^{n} b_j.$$

Using this result, show that the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges for every complex number $z$ with $|z| = 1$, and $z \neq 1$.

5. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n(n + 1)}$$

converges for all $|z| \leq 1$. What is its radius of convergence? Does the series

$$\sum_{n=1}^{\infty} z^n$$

converge for any $z$ with $|z| = 1$?
6. Let $n$ be a non-negative integer. Define for $\alpha \in \mathbb{C}$, the binomial coefficient
\[
\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}.
\]
Show that the series
\[
\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n
\]
converges for $|z| < 1$.

7. Calculate the line integral
\[
\int_{L} \frac{dz}{z}
\]
where $L$ is the arc of the unit circle from 1 to $i = \sqrt{-1}$ traversing in the counterclockwise direction. More generally, what is the answer if $i$ is replaced with $e^{i\theta}$ for some $\theta$ satisfying $0 \leq \theta \leq 2\pi$.

8. Compute
\[
\int_{C} \frac{\sin z}{z^2} dz
\]
where $C$ is the circle of radius 1 centered at zero and oriented clockwise. What is the answer if $z^2$ is replaced with $z^3$ in the integral?

9. Show that
\[
\max_{|z| \leq 1} |e^{z^2}| = e.
\]

10. If $\alpha, \beta, \gamma$ are the angles of a triangle and $a, b, c$ are the lengths of the corresponding opposite sides, show that
\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{d},
\]
where $d$ is the diameter of the circumscribed circle of the triangle.