

# MATH 497/812: Assignment 1

Due: October 11, 2011

Math 497: Do any eight questions.

Math 812: Do all questions.

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1. Let  $\chi_4(n)$  be defined as follows:  $\chi_4(n) = 0$  if  $n$  is even,  $+1$  if  $n \equiv 1 \pmod{4}$  and  $-1$  if  $n \equiv 3 \pmod{4}$ . Show that  $\chi_4(mn) = \chi_4(m)\chi_4(n)$ .
2. With  $\chi_4$  defined in the previous question and  $r_2(n)$  denoting the number of ways  $n$  can be written as a sum of two squares, show that

$$r_2(n) = 4 \sum_{d|n} \chi_4(d).$$

3. Define  $\tau(n)$  be

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Prove that  $\tau(n)$  is odd if and only if  $n = (2m + 1)^2$  for some  $m$ . [Hint: Reduce the product defining  $\tau(n)$  modulo 2.]

4. Let  $r_k(n)$  be the number of ways of writing  $n$  as a sum of  $k$  squares. Show that

$$r_k(n) = \sum_{a=1}^n r_i(a) r_{k-i}(n - a),$$

for any  $i$  satisfying  $1 \leq i \leq k$ .

5. Show that  $SL_2(\mathbb{Z})$  is generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

[Hint: Compute  $T^{-1}UT^{-1}$ .]

6. Show that  $SL_2(\mathbb{Z}/N\mathbb{Z})$  is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and that both of these elements have order  $N$ .

7. Prove the following variation on the division algorithm. Given natural numbers  $a, b$  with  $b \neq 0$ , there are integers  $q$  and  $r$  such that  $a = bq + r$  with  $|r| \leq b/2$ .

8. Show that  $\Gamma(2)$  is generated by  $-I, T^2$  and  $U^2$  where

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

9. (a) If  $G$  is a group which acts transitively on a set  $X$ , show that all stabilizers are conjugate.
- (b) Let  $G$  be a group and  $H, K$  subgroups. Say that  $x, y \in G$  are equivalent if there exist  $h \in H$  and  $k \in K$  such that  $x = hyk$ . If  $x$  and  $y$  are equivalent, we write  $x \sim y$ . Show that  $\sim$  is an equivalence relation.
10. (a) If  $\Gamma$  is a congruence subgroup of level  $N$  and  $\gamma$  is in  $SL_2(\mathbb{Z})$ , show that  $\gamma\Gamma\gamma^{-1}$  is again a congruence subgroup of level  $N$ .
- (b) Prove that  $\Gamma(M) \cap \Gamma(N) = \Gamma(\text{lcm}(M, N))$ .
- (c) Prove that  $\Gamma(M)\Gamma(N) = \Gamma(\text{gcd}(M, N))$ .