MATH 497/812: Assignment 2

Due: October 25, 2011

Math 497: Do any eight questions.

Math 812: Do all questions.

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1. Show that $PSL_2(\mathbb{R})$ acts faithfully on the upper half-plane. That is, if $gz = z$ for all z in the upper half-plane, then $g = 1$.

2. Show that the map $z \mapsto (z - i)/(z + i)$ is an analytic isomorphism of the upper half plane onto the unit disc consisting of complex numbers z with $|z| < 1$.

3. If Γ is of finite index in $SL_2(\mathbb{Z})$, show that the number of Γ-inequivalent cusps is $\leq [SL_2(\mathbb{Z}) : \Gamma]$.

4. Show that $\{0, i\infty\}$ is a complete set of inequivalent cusps for $\Gamma_0(2)$.

5. Let p be a prime number. Show that the matrices
$$\begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}, \quad 0 \leq k \leq p - 1,$$
along with the identity, gives a complete set of right coset representatives for $\Gamma_0(p)$ in $SL_2(\mathbb{Z})$.

6. Let p be prime. Show that a complete set of inequivalent cusps for $\Gamma_0(p)$ is given by $\{0, i\infty\}$.

7. If Γ is a congruence subgroup of $SL_2(\mathbb{Z})$ and $\mathcal{D}$ is a fundamental domain for Γ in the upper half-plane, show that the volume of $\mathcal{D}$ is equal to
$$[SL_2(\mathbb{Z}) : \Gamma]\frac{\pi}{3}.$$

8. Let
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}),$$
and z lie in the upper half-plane. Define $j(\gamma, z) := cz + d$. Show that
$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)j(\gamma_2, z)$$
for any $\gamma_1, \gamma_2 \in GL_2^+(\mathbb{R})$.

9. (a) Let B_k denote the k-th Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_k x^k}{k!}.$$

Show that $B_k = 0$ for k odd and greater than 1.

(b) Show that for k even, $B_k < 0$ if and only if $4|k$.

10. Prove that if f is a modular form of weight k for the full modular group and has integer Fourier coefficients, then f can be written as a polynomial in E_4, E_6 and Δ with integer coefficients.