MATH 497/812: Assignment 3

Due: November 18, 2011

Math 497: Do any eight questions.

Math 812: Do all questions.

---

1. Let E_k denote the normalized Eisenstein series of weight k for $SL_2(\mathbb{Z})$. Show that $E_4^2 = E_8$ and $E_4E_6 = E_{10}$.

2. Let $j(z) = E_4^3(z)/\Delta(z)$. Show that the map $z \to j(z)$ defines a bijection from the standard fundamental $\mathcal{F}$ for $SL_2(\mathbb{Z})$ onto the complex plane $\mathbb{C}$.

3. For z in the upper half-plane, define the Dedekind η-function by

$$\eta(z) = e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).$$

   Show that

$$\eta(-1/z) = \sqrt{\frac{z}{i}}\, \eta(z),$$

   where the square root is the branch with non-negative real part. [Hint: Take logarithmic derivative and compare with E_2.]

4. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. The *Serre derivative* $\vartheta_k(f)$ of a modular form $f \in M_k(\Gamma)$ is defined as

$$\vartheta_k(f)(z) := \frac{1}{2\pi i} f'(z) - \frac{k}{12} E_2(z) f(z).$$

   Show that ϑ_k maps $M_k(\Gamma)$ into $M_{k+2}(\Gamma)$.

5. Let $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ with $q = e^{2\pi iz}$ be a cusp form of weight k for the full modular group which is also an eigenform for all the Hecke operators T_m, $m \geq 1$. If p is a prime, show that

$$\sum_{n=0}^{\infty} a(p^n) x^n = (1 - a(p)x + p^{k-1}x^2)^{-1}.$$

6. Let $f(z) = \sum_{n=1}^{\infty} a(n) q^n$, with $q = e^{2\pi iz}$ be a cusp form of weight 26 for the full modular group. If $a(1) = 1$, show that $a(mn) = a(m)a(n)$ whenever m, n are relatively prime.

7. Let $\tau(n)$ denote the Ramanujan τ-function and $\sigma(n)$ be the sum of the positive divisors of n. Show that

$$(1-n)\tau(n) = 24\sum_{j=1}^{n-1}\sigma(j)\tau(n-j).$$

8. (a) If $d(n)$ is the number of divisors of n and $n = p_1^{a_1}\cdots p_k^{a_k}$ is the unique factorization of n into distinct prime powers, show that $d(n) = (a_1+1)\cdots(a_k+1)$.

   (b) Assuming the truth of Ramanujan's conjecture for each prime p, write $\tau(p) = 2p^{11/2}\cos\theta_p$. Using induction on m, prove that

$$p^{-\frac{11m}{2}}\tau(p^m) = \frac{\sin(m+1)\theta_p}{\sin\theta_p}.$$

   Deduce that $|\tau(n)| \leq n^{\frac{11}{2}}d(n)$, where $d(n)$ denotes the number of divisors of n.

9. Suppose that f is a modular form of weight k for the full modular group with Fourier expansion at infinity given by

$$\sum_{n=0}^{\infty} a(n)q^n, \quad q = e^{2\pi iz}.$$

   Show that the series

$$L(s) := \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

   extends to an analytic function for all complex values of s, except (possibly) a simple pole at $s=0$ and $s=k$ and satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L(s) = i^k(2\pi)^{-(k-s)}\Gamma(k-s)L(k-s).$$

10. If $f(z)$ is a modular form of weight k for the full modular group, show that $f(Nz)$ is a modular form of weight k for $\Gamma_0(N)$.