SPECIAL VALUES OF THE POLYGAMMA FUNCTIONS

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Abstract. Let q be a natural number and $f : \mathbb{Z}/q\mathbb{Z} \to \overline{\mathbb{Q}}$. We consider the Dirichlet series $\sum_{n\geq 1} f(n)/n^s$ and relate its value when s is a natural number, to the special values of the polygamma function. For certain types of functions f, we evaluate the special value explicitly and use this to study linear independence over $\mathbb{Q}$ of $L(k,\chi)$ as χ ranges over Dirichlet characters mod q which have the same parity as k.

1. Introduction

The *digamma* function $\psi(x)$ is the logarithmic derivative of the Gamma function, $\Gamma(x)$. In an earlier paper [12], we investigated special values of the digamma function. Let $q > 1$ be a fixed natural number and let γ and $\varphi(q)$ denote Euler's constant and the Euler φ-function respectively. We showed [12] that all of the $\varphi(q)+1$ numbers,

$$\gamma, \psi(a/q), \quad (a,q)=1, 1 \leq a < q,$$

are transcendental, with at most one exception. Our methods led to an interesting corollary: all the numbers $\psi(a/q)+\gamma$ with $1 \leq a < q$ and $(a,q)=1$, are transcendental. (This last result was obtained earlier by Bundschuh [4] using a completely different approach.) In particular, if this fugitive exception alluded to above exists among the $\psi(a/q)$'s, then γ is transcendental. For various reasons discussed in [12], it is highly unlikely that the exception exists. Our method and motivation allowed us to obtain a species of results. We conjectured that if K is an algebraic number field over which the q-th cyclotomic polynomial is irreducible, then the $\varphi(q)$ numbers

$$\psi(a/q), \quad (a,q)=1, \quad 1 \leq a < q$$

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are linearly independent over K. If K is as given, we proved that the numbers $\psi(a/q)+\gamma$ are linearly independent over K and the K-vector space spanned by γ and the $\varphi(q)$ numbers $\psi(a/q)$ with $(a,q)=1$ and $1 \leq a < q$, has dimension at least $\varphi(q)$. Our method was motivated by a fundamental paper of Baker, Birch and Wirsing [3] who resolved a conjecture of Chowla concerning the vanishing of certain Dirichlet series. This method leads naturally to considerable generalization and one is led to enquire about special values of Chowla-type Dirichlet series at other integral points.

In this paper, we will study further aspects of the digamma function as well as the special values of the polygamma functions $\psi_k(x)$ defined as the k-th derivative of $\psi(x)$. Thus, $\psi_0(x)$ is the digamma function $\psi(x)$. These functions arise naturally when one studies special values of the Hurwitz zeta function.

More precisely, given a function $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, one can study the Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

If $\zeta(s,x)$ denotes the Hurwitz zeta function

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

we may write

$$L(s,f) = q^{-s} \sum_{a=1}^{q} f(a)\zeta(s,a/q). \tag{1}$$

Since $\zeta(s,x)$ admits an analytic continuation to the entire complex plane apart from a simple pole at $s=1$ with residue 1, we deduce that $L(s,f)$ extends to the entire complex plane except at $s=1$ where it has a simple pole with residue

$$\frac{1}{q}\sum_{a=1}^{q} f(a). \tag{2}$$

Thus, the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \tag{3}$$

converges if and only if (2) vanishes. In this case, one can show (see [12]) that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q} \sum_{a=1}^{q} f(a)\psi_0(a/q). \tag{4}$$

In 1973, Baker, Birch and Wirsing [3], in response to a question raised by S. Chowla [6], proved that (3) is non-zero if f takes algebraic values with period q and $f(r) = 0$ whenever $1 < (r,q) < q$ and the q-th cyclotomic polynomial is irreducible over the field generated by the values of f. (This irreducibility criterion can be stated another way. If f takes values in an algebraic number field K, then the irreducibility over K of the q-th cyclotomic polynomial is equivalent to $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. A proof can be found in [12]. This, in turn, can be deduced by an application of Theorem 29 of [2] applied to our context.) In [12], we were able to utilise this result together with judicious choices of the function f to deduce various transcendence results of the digamma function at rational arguments. In this paper, we expand on this theme. More precisely, we consider $L(s,f)$ for s equal to a positive integer. This will allow us to deduce various transcendence results concerning the polygamma function.

The polygamma function has several familiar incarnations. Indeed, we have

$$-\psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right),$$

so that for $k \geq 1$, we have

$$\psi_k(z) = (-1)^{k-1} k! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}}.$$

For future reference, let us note that

$$\psi_k(z+1) = \psi_k(z) + \frac{(-1)^k k!}{z^{k+1}}. \tag{5}$$

Thus, our focus of attention is really about special values of the Hurwitz zeta function. It is now a direct consequence of (1) that (4) can be generalized to

$$L(k,f) = \frac{(-1)^k}{(k-1)!q^k} \sum_{a=1}^{q} f(a)\psi_{k-1}(a/q). \tag{6}$$

The polygamma function is also related to the cotangent function in the following way. Recall that for z not an integer,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

Thus, if k is odd, we deduce that

$$-\frac{d^k}{dz^k} (\pi \cot \pi z) = \psi_k(z) + \psi_k(-z) - \frac{k!}{z^{k+1}}.$$

If k is even, we have

$$-\frac{d^k}{dz^k} (\pi \cot \pi z) = \psi_k(z) - \psi_k(-z) + \frac{k!}{z^{k+1}}.$$

Thus, from (5), we have

$$-\frac{d^k}{dz^k} (\pi \cot \pi z) = \psi_k(z) + (-1)^{k+1} \psi_k(1-z). \tag{7}$$

A natural generalization of Chowla's question is to determine when $L(k, f)$ vanishes. The case $k = 1$ was treated in several papers [3], [6], and [14]. We may seek to classify those integer valued functions f for which $L(k, f) = 0$. To this end, let us note the following example. Suppose that $f(a) = m$ for some fixed integer m for all a satisfying $1 \leq a \leq q-1$. Then,

$$L(k, f) = \frac{m}{q^k} \sum_{a=1}^{q-1} \zeta(k, a/q) + \frac{f(q)\zeta(k)}{q^k}.$$

Since

$$(1 - q^{-s})\zeta(s) = \frac{1}{q^s} \sum_{a=1}^{q-1} \zeta(s, a/q),$$

we deduce that

$$L(k, f) = \zeta(k) \left(\frac{f(q) + m(q^k - 1)}{q^k} \right).$$

This means that if we define $f(q) = -m(q^k - 1)$, then $L(k, f) = 0$. Are there other such examples? In [5], P. and S. Chowla proposed a conjecture that this is the only example in the special case $k = 2$ and q an odd prime. More precisely, they conjectured that if f is rational valued, then $L(2, f) = 0$ if and only if there is a non-zero rational number m such that $f(a) = m$ for $1 \leq a \leq q-1$ and $f(q) = -m(q^2-1)$ for q an odd prime. In particular, this conjecture implies that the numbers $\psi_1(a/q)$ for $1 \leq a \leq q-1$ and q an odd prime, are linearly independent over $\mathbb{Q}$. In fact, one can show that this is equivalent to the Chowlas' conjecture (see for example, page 300 of [10]). In case $q = 2$, it is easy to see that $\psi_1(1/2) = \pi^2/2$, which is transcendental. Also, we deduce from (7) that for any a coprime to q, with $1 \leq a < q$, at least one of $\psi_k(a/q), \psi_k(1-a/q)$ is transcendental. Indeed, the left hand side of (7) evaluated at $z = a/q$ is an algebraic multiple of π^{k+1}. It is a non-zero algebraic multiple by virtue of Lemma 9. More generally, we formulate the following.

Conjecture 1. *Let K be an algebraic number field over which the q-th cyclotomic polynomial is irreducible. Then, the $\varphi(q)$ numbers, $\psi_k(a/q)$ with $1 \leq a \leq q$ and $(a,q) = 1$, are linearly independent over K.*

We will explore the consequences of this conjecture in later sections. For instance, the conjecture implies that at most one of the $\varphi(q)$ numbers

$$L(k,\chi), \quad \chi \,(\mathrm{mod}\, q),$$

is rational if $(\varphi(q), q) = 1$. Let us note that for the trivial character, $L(k,\chi)$ is a rational multiple of $\zeta(k)$. It is reasonable to conjecture that all of the numbers $\psi_k(a/q)$ are transcendental. Perhaps it is even true that $L(k,f)$ is transcendental whenever it is non-zero and f takes algebraic values. However, this seems beyond reach at present since there is no satisfactory extension of Baker's theorem to linear forms in polylogarithms. In section 4, we derive some cases in which we can assert that $L(k,f)$ is transcendental. In the case $k = 1$, the fact that $L(1,f)$ is transcendental whenever it is non-zero was proved in [1] using Baker's theory of linear forms in logarithms.

In the context of this conjecture, it seems important to single out those algebraic valued functions f with period q which are supported only on the coprime residue classes mod q. We will say that such an f is of *Dirichlet type* since it is necessarily a linear combination of Dirichlet characters. The conjecture would imply that for such a function f, whose values generate a field K that satisfies $K \cap \mathbb{Q}(\zeta_q)$, we always have $L(k,f) \neq 0$. We will study these functions in later sections.

2. Polylogarithms

The polylogarithm function $Li_k(z)$ is defined by

$$Li_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Thus, $Li_1(z) = -\log(1-z)$. We will show below that the polylogarithms are related to the polygamma functions.

Let f be a periodic function with period q. We define its *Fourier transform* $\hat{f}$ by

$$\hat{f}(n) = \frac{1}{q}\sum_{a=1}^{q} f(a)e^{2\pi i a n/q}.$$

We see that $\hat{f}$ is also periodic with period q. Moreover, $\hat{f}$ is even or odd according as f is even or odd. By orthogonality, we have

$$f(n) = \sum_{a=1}^{q} \hat{f}(a)e^{-2\pi i a n/q}.$$

Thus, we may write

$$L(k,f) = \sum_{n=1}^{\infty}\frac{1}{n^k}\sum_{a=1}^{q}\hat{f}(a)e^{-2\pi i a n/q} = \sum_{a=1}^{q}\hat{f}(a)Li_k(e^{-2\pi i a/q}). \tag{8}$$

If $\hat{f}(q) = 0$, let us note that this is also valid for $k = 1$. In particular,

$$2L(k,f) = \sum_{a=1}^{q}\hat{f}(a)\left(Li_k(e^{2\pi i a/q}) \pm Li_k(e^{-2\pi i a/q})\right),$$

assuming that f is an even or odd function. It is well-known that the k-th Bernoulli polynomial has rational coefficients and for $0 < x < 1$, $B_k(x)$ has the Fourier series expansion

$$B_k(x) = -\frac{k!}{(2\pi i)^k}\sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n x}}{n^k},$$

where the dash on the summation indicates that $n = 0$ is omitted from the sum. This means that

$$-B_k(x) = \frac{k!}{(2\pi i)^k}[(-1)^k Li_k(e^{-2\pi i x}) + Li_k(e^{2\pi i x})].$$

It is convenient to say that f and k have the *same parity* whenever f is an even function and k is an even integer or when f is an odd function and k is an odd integer. With this understanding, we can put the above discussion in the following convenient form:

Proposition 2. *Let f be a periodic function with period q. If k and f have the same parity, and $k > 1$, then*

$$2L(k,f) = (-1)^{k-1}\frac{(2\pi i)^k}{k!}\sum_{a=1}^{q}\hat{f}(a)B_k(a/q).$$

Thus, if f takes algebraic values, then it is an algebraic multiple of π^k. If in addition $L(k,f)$ is non-zero, then it is transcendental.

The non-vanishing of $L(1,f)$ when f is of Dirichlet type will be taken up in the next section.

3. Functions of Dirichlet type

Recall that an algebraic valued function $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ is be said to be of *Dirichlet type* if $f(a) = 0$ whenever $(a,q) > 1$. Clearly, such a function can be written as a linear combination of Dirichlet characters mod q.

We recall the following result (see p. 14 of [11]) which will be used later, for the convenience of the reader.

Lemma 3. *Let $q \geq 2$ and $\Phi_q(x)$ denote the q-th cyclotomic polynomial. Then $\log \Phi_q(1) = \Lambda(q)$, where $\Lambda(q)$ is the von Mangoldt function. In particular, $\Phi_q(1)$ is 1 when q is not a prime power.*

Proof. Since

$$x^q - 1 = \prod_{d|q} \Phi_d(x),$$

we have

$$\frac{x^q - 1}{x - 1} = \prod_{d|q, d>1} \Phi_d(x).$$

Taking log of the limits as $x \to 1$, we obtain

$$\log q = \sum_{d|q, d>1} \log \Phi_d(1).$$

Since

$$\log q = \sum_{d|q} \Lambda(d),$$

the result is now immediate by Möbius inversion. □

We would like to describe those functions f such that $\hat{f}$ is of Dirichlet type and for which $L(1,f) = 0$. By our earlier remarks, we need to assume that (2) vanishes to ensure convergence of the series. This will be tacitly assumed in the discussion below. Let us define f_h by setting $f_h(n) := f(hn)$. In case f is rational valued, or if f assumes values in an algebraic number field K with $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, one can deduce (as in [3]) that $L(1, f_h) = 0$ for every h coprime to q whenever $L(1,f) = 0$. (See Corollary 5 below.) This raises the natural question of characterizing those functions f with period q such that $L(1, f_h) = 0$ for every h coprime to q. We will show below that if q is a prime power, then no such non-trivial function exists. If q is not a prime power, however, then one such function is given by the *Ramanujan sum* $c_q(n)$ defined as

$$c_q(n) = \sum_{(a,q)=1} e^{2\pi i a n/q},$$

which is a multiplicative function of n and for which we have the explicit formula (see for example, page 7 of [11])

$$c_q(n) = \mu(q/n_1)\varphi(q)/\varphi(q/n_1),$$

where $n_1 = (n, q)$. By definition, $c_q(n)$ is the Fourier transform of the function $q\delta(a)$ which is q whenever $(a, q) = 1$ and zero otherwise. (We see that $\hat{c}_q$ is of Dirichlet type but c_q itself is not.) Therefore, $\hat{c}_q(a) = 1$ if $(a, q) = 1$ and zero otherwise, and we have by (8)

$$\sum_{n=1}^{\infty} \frac{c_q(n)}{n} = -\sum_{(a,q)=1} \log(1 - e^{-2\pi i a/q}) = -\log \Phi_q(1), \quad \text{for } q \geq 2,$$

where Φ_q denotes the q-th cyclotomic polynomial. But by Lemma 3, the sum is zero (see also p. 14 of [11]). Our claim is that this is the only example. More precisely, we have:

Theorem 4. *Assume that f is an algebraic valued periodic function with period q and not identically zero. Let $f_h(n) = f(hn)$ and suppose that $\hat{f}$ is of Dirichlet type. If q is a prime power, then $L(1, f_h) \neq 0$, for some h coprime to q. If q is not a prime power, and f is not a scalar multiple of c_q, then $L(1, f_h) \neq 0$ for some h coprime to q.*

Before we proceed to the proof of this assertion, we review some basic facts about Dedekind determinants.

Let A be a finite abelian group and $f : A \to \mathbb{C}$ any complex valued function on A. The determinant of the $|A| \times |A|$ matrix given by $(f(xy^{-1}))$ as x, y range over the elements of A is usually called the Dedekind determinant (see [9]) and is known to equal

$$\prod_{\chi} \left(\sum_{x \in A} \chi(x) f(x) \right),$$

where the product is over all the characters of A. One can be more precise. The eigenvalues are given by

$$\sum_{a \in A} f(a)\chi(a)$$

as χ ranges over the irreducible characters of A. The dimension of the eigenspace corresponding to the eigenvalue 0 is equal to the number of these eigenvalues which are equal to zero. With these preliminary remarks, we proceed to the proof of the following.

Proof of Theorem 4. Suppose that $L(1, f_h) = 0$ for every h coprime to q. By (8), this means

$$L(1, f_h) = -\sum_{a=1}^{q} \hat{f}(a) \log(1 - e^{-2\pi i a h/q}) = 0. \tag{9}$$

This should be viewed as a matrix equation. Since $\hat{f}$ is of Dirichlet type, the summation is over the coprime residue classes mod q. If we let B be the $\varphi(q) \times \varphi(q)$ matrix whose (a, h)-th entry is given by

$$-\log(1 - e^{-2\pi i a h/q}),$$

then by the evaluation of the Dedekind determinant, we see that it is (upto sign)

$$(-\log \Phi_q(1)) \prod_{\chi \neq \chi_0} L(1, \chi).$$

First suppose that q is a prime power. By Lemma 3 (see p. 14 of [11]) we have that $\Phi_q(1) \neq 1$ in this case. Thus, the matrix B is invertible and we deduce that $\hat{f}$ is identically zero. By Fourier inversion, this means f is identically zero, contrary to assumption. Now let us consider the case when q is not a prime power. In this case, the matrix B has eigenvalue zero and the corresponding eigenspace is one-dimensional since all the other $\varphi(q) - 1$ eigenvalues are non-zero, being the values $L(1, \chi)$ as χ ranges over non-trivial Dirichlet characters mod q. As seen earlier, $\hat{c}_q = \chi_0$. So, it is an element of this space, and any other element in this space must be a scalar multiple of it. The equation (9) implies that the column vector $\hat{f}(a)$, $1 \leq a \leq q, (a, q) = 1$ is in the zero eigenspace. Thus, f is a scalar multiple of the Ramanujan sum. □

Corollary 5. *Let K be an algebraic number field with $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Assume that f is a K-valued periodic function with period q and not identically zero and that $\hat{f}$ is of Dirichlet type. If q is a prime power, then $L(1, f) \neq 0$. If q is not a prime power, then $L(1, f) \neq 0$ unless $f = \lambda c_q$ for some $\lambda \in K$.*

Proof. As in [3], we can deduce that if $L(1, f) = 0$ and f takes values in an algebraic number field K with $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, then $L(1, f_h) = 0$ for every h coprime to q. Since $\hat{f}$ is of Dirichlet type, Theorem 4 implies that f is identically zero if q is a prime power and a scalar multiple of c_q if q is not a prime power. □

These results can be viewed as variations of the theme investigated in [3] regarding Chowla's problem.

Tijdeman [16] proved that if f is a rational valued completely multiplicative function with period q, then $L(1,f) \neq 0$. We derive a variant of this result from Theorem 4.

Corollary 6. *Let f be an algebraic-valued completely multiplicative function with period q, not identically zero, such that $\hat{f}$ is of Dirichlet type. Then $L(1,f) \neq 0$.*

Proof. Since f is completely multiplicative, we see that $f_h(n) = f(h)f(n)$. Hence, if $L(1,f) = 0$, then $L(1,f_h) = 0$ for every h. If q is a prime power, then we have a contradiction by Theorem 4. If q is not a prime power, we must have that f is a scalar multiple of c_q, by the same theorem. Since $f(1)^2 = f(1)$, the scalar can only be zero or 1. The former case is ruled out since f is not identically zero. The latter case is also ruled out since the Ramanujan sum, $c_q(n)$ is not completely multiplicative. □

In their study of periodic functions f for which $L(1,f) = 0$, Baker, Birch and Wirsing [3] defined the class F_q of all algebraic valued functions f with period q such that $L(1,f) = 0$. They proved the following lemma.

Lemma 7. *If $f \in F_q$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and h is an integer defined mod q by $\sigma^{-1}e^{2\pi i/q} = e^{2\pi i h/q}$, then the function $f'(n) := \sigma(f(hn))$ is also in F_q.*

Theorem 8. *Let K be an algebraic number field and $K_1 = K(e^{2\pi i/\varphi(q)})$. Suppose that $K_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then, the values $L(1,\chi)$, as χ ranges over non-trivial primitive characters mod q, are linearly independent over K_1.*

Remark. By the theorem, any non-trivial K_1-linear combination Λ of the values $L(1,\chi)$ with χ non-trivial and primitive (mod q) is non-zero. On the other hand, each $L(1,\chi)$ is a $\overline{\mathbb{Q}}$-linear combination of logarithms of algebraic numbers. By Baker's theorem, we deduce that Λ is transcendental. In [3], Baker, Birch and Wirsing have shown that the numbers $L(1,\chi)$ as χ ranges over all non-trivial characters, are linearly independent over $\mathbb{Q}$ under the condition that $(q,\varphi(q)) = 1$. It is believed, but not yet proved, that this is still true without this latter coprimality condition.

Proof. Suppose that

$$\sum_{\chi \neq \chi_0} c_\chi L(1,\chi) = 0, \tag{10}$$

where the summation is over primitive Dirichlet characters and the c_χ are in K_1. Then, define

$$f = \sum_{\chi \neq \chi_0} c_\chi \chi.$$

This is clearly of Dirichlet type. If f is identically zero, then by orthogonality of characters, we deduce $c_\chi = 0$ for all $\chi \neq \chi_0$. So we may suppose that f is not identically zero. Moreover, by the theory of Gauss sums (see [7]), we have for any primitive character $\chi \neq \chi_0$,

$$\hat{\chi}(n) = \frac{\tau(\chi)\overline{\chi}(n)}{q},$$

with $|\tau(\chi)| = 1$ (see for example, exercises 5.3.1, 5.3.2 and 5.3.3 in [11]). Thus, if χ is a primitive Dirichlet character $\neq \chi_0$, then $\hat{\chi}$ is of Dirichlet type. Hence, $\hat{f}$ is also of Dirichlet type and we have

$$\hat{f} = \frac{1}{q} \sum_{\chi \neq \chi_0} c_\chi \tau(\chi) \overline{\chi}. \tag{11}$$

By Lemma 7, there is a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ so that $f'(n) := \sigma(f(hn)) \in F_q$ for every $(h,q) = 1$. The values of f lie in the field K_1 which is disjoint with the q-th cyclotomic field. Thus, $\sigma(f(hn)) = f(hn)$ for every $(h,q) = 1$ implying that $f_h \in F_q$. This means that $L(1, f_h) = 0$ for every $(h,q) = 1$. We may apply Theorem 4. If q is a prime power, $L(1,f) \neq 0$, and we have a contradiction to (10). If q is not a prime power, then f must be scalar multiple of the Ramanujan sum. But this means that $\hat{f}$ must be a non-zero scalar multiple of the trivial character. By (11) and the orthogonality of characters, we deduce that $c_\chi = 0$ for all $\chi \neq \chi_0$. The second assertion follows by an application of Baker's theorem on linear forms of logarithms. □

We would like to assert a similar theorem for $L(k, \chi)$.

4. Even and odd functions of Dirichlet type

It is possible to make some remarks about the transcendence of $L(k,f)$ when f is an even or odd function of Dirichlet type. A key role will be played by the following lemma due to Okada [13].

Lemma 9. (Okada [13]) *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representatives mod q such that the union $T \cup (-T)$ is a complete set of residues prime to q. Then, the real numbers*

$$\frac{d^{k-1}}{dz^{k-1}}(\cot \pi z)\Big|_{z=a/q},$$

$a \in T$, are linearly independent over $\mathbb{Q}$.

We remark that the cases $k = 1$ and 2 of the above lemma were proved earlier by Chowla [6] and Jager and Lenstra [8].

In the discussion below, we will take $q \geq 3$ and T to be

$$\{1 \leq a \leq q/2 : (a, q) = 1\}.$$

Thus, the set

$$\{a, q - a : a \in T\},$$

is a complete set of coprime residue classes (mod q).

We will use this to show:

Theorem 10. *Let f be of Dirichlet type, rational valued and not identically zero. If f and k have the same parity, then $L(k, f)$ is a non-zero algebraic multiple of π^k.*

Proof. We have

$$L(k, f) = \frac{(-1)^k}{(k-1)!q^k} \sum_{a=1}^{q} f(a)\psi_{k-1}(a/q).$$

When f and k have the same parity, this yields

$$L(k, f) = \frac{(-1)^k}{(k-1)!q^k} \sum_{a \in T} f(a)(\psi_{k-1}(a/q) \pm \psi_{k-1}(1 - a/q)),$$

according as f and k are both even or odd. Now use (7) to get

$$L(k, f) = -\frac{(-1)^k}{(k-1)!q^k} \sum_{a \in T} f(a) \left(\frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z) \Big|_{z=a/q} \right)$$

By Lemma 9, the sum is non-zero. Also, Proposition 2 implies that it is an algebraic multiple of π^k. $\square$

In particular, by taking $q = 4$, $f(1) = 1$ and $f(3) = -1$, this series is the well-known formula of Nilakantha of the 15th century (later rediscovered by Gregory and Leibniz, see [15]):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Taking $k = 3$ in our previous theorem, another well-known formula appears:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

A careful examination of the proof of Lemma 9 given in [13] shows that the following general result is true.

Lemma 11. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representatives mod q such that the union $T \cup (-T)$ is a complete set of coprime residues mod q. Let K be an algebraic number field over which the q-th cyclotomic polynomial is irreducible. Then, the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}}(\cot \pi z)\Big|_{z=a/q}, \quad a \in T$$

is linearly independent over K.

Proof. Since this is not explicitly stated in [13], we will indicate the necessary observations in [13] to deduce the result. First, we have already noted in [12] that the irreducibility of the q-th cyclotomic polynomial over K is equivalent to $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. As in [13], let us set for x not an integer,

$$F_k(z) = \frac{k}{(-2\pi i)^k} \frac{d^{k-1}}{dz^{k-1}}(\pi \cot \pi z).$$

Equation (5) of [13] shows that $F_k(b/q) \in \mathbb{Q}(\zeta_q)$, where ζ_q is a primitive q-th root of unity. We now suppose that

$$\sum_{b \in T} C_b F_k(b/q) = 0,$$

with $C_b \in K$ and follow the argument on pages 343 and 344 of [13]. Since K is disjoint from $\mathbb{Q}(\zeta_q)$, the mappings σ_a $(a \in T)$ fix C_b and so the equation

$$\sum_{b \in T} C_b F_k(\overline{a} b/q) = 0,$$

holds (with notation as in [13]). The remainder of the proof is valid *mutatis mutandis* and we deduce that $C_b = 0$ for all $b \in T$. □

We remark that Okada [13] seems to be aware of this generalization of his theorem since he makes use of it in his two corollaries at the end of the paper. This generalization allows us to deduce:

Theorem 12. *Let K be an algebraic number field with $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Let f be a function of Dirichlet type with values in K, not identically zero. If f and k have the same parity, then $L(k, f)$ is a non-zero algebraic multiple of π^k and hence transcendental.*

5. Linear independence of special values of Dirichlet L-functions

Theorem 13. *Let K be an algebraic number field and $K_1 = K(e^{2\pi i/\varphi(q)})$. Suppose that $K_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Fix a natural number k. Then, the set of numbers $L(k,\chi)$ as χ ranges over Dirichlet characters mod q with the same parity as k are linearly independent over K_1.*

Proof. Now suppose that

$$\sum_{\chi \sim k} c_\chi L(k,\chi) = 0, \quad c_\chi \in K_1,$$

where the sum is over those characters χ with the same parity as k. The function

$$f = \sum_{\chi \sim k} c_\chi \chi$$

also has the same parity as k and takes values in K_1. By Theorem 12, we deduce that f is identically zero. By orthogonality of characters, we find that $c_\chi = 0$ for all χ, which is a contradiction. This completes the proof. □

We remark that if Conjecture 1 is true, then Theorem 12 can be generalized as follows: if f is a non-zero function of Dirichlet type taking values in an algebraic number field K and $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, then $L(k,f) \neq 0$. An immediate consequence of this would be the linear independence over K of $L(k,\chi)$ as χ ranges over Dirichlet characters mod q when $(q, \varphi(q)) = 1$.

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