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Ramanujan Graphs

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In the last two decades, the theory of Ramanujan graphs has gained prominence primarily for two reasons. First, from a practical viewpoint, these graphs resolve an extremal problem in communication network theory (see for example [2]). Second, from a more aesthetic viewpoint, they fuse diverse branches of pure mathematics, namely, number theory, representation theory and algebraic geometry. The purpose of this survey is to unify some of the recent developments and expose certain open problems in the area. This survey is by no means an exhaustive one and demonstrates a highly number-theoretic bias. For more comprehensive surveys, we refer the reader to [27], [9] or [13]. For a more up-to-date survey highlighting the connection between graph theory and automorphic representations, we refer the reader to Winnie Li's recent survey article [11].

A *graph* X is a triple consisting of a *vertex set* $V = V(X)$, an *edge set* $E = E(X)$ and a map that associates to each edge two vertices (not necessarily distinct) called its *endpoints*. A *loop* is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. A *simple graph* is one having no loops or multiple edges. If a graph has loops or multiple edges, we will call it a *multigraph*. When two vertices u and v are endpoints of an edge, we say they are *adjacent* and (sometimes) write $u \sim v$ to indicate this. To any graph, we may associate the *adjacency matrix* A which is an $n \times n$ matrix (where $n = |V|$) with rows and columns indexed by the elements of the vertex set and the (x, y)-th entry is the number of edges connecting x and y. Since our graphs are undirected, the matrix A is symmetric. Consequently, all of its eigenvalues are real.

The convention regarding terminology is not clear in the literature. Most use the term 'graph' to mean a simple graph as we have defined it above. Thus, the

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Ramanujan graphs constructed in the literature, which we shall refer to below, have been all simple graphs. One can enlarge the definition of a Ramanujan graph to the context of multigraphs so that their construction becomes slightly simpler (see for instance, Theorem 8 below and the example that follows).

The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges incident with v, where we count a loop with multiplicity 2. With this convention, we have the familiar "handshaking lemma" of graph theory:

$$\sum_{v \in V} \deg(v) = 2|E(X)|.$$

We will say an edge e has *length* 1, unless it is a loop, in which case we adopt the convention that it has length 2. We will denote the length of e by $\ell(e)$. For a multigraph, a *walk* of *length* r from x to y is a sequence $x = v_0, v_1, ..., v_r = y$ with $v_i \in V$ and $e_i = (v_i, v_{i+1}) \in E$ for $i = 0, 1, ..., r - 1$ and

$$\sum_i \ell(e_i) = r.$$

A *path* is a walk with no repeated vertex. A word of caution must be inserted here. In graph theory literature, the distinction between a walk and a path is as we have defined it above. However, in the number theory circles, the finer distinction is not made and one uses the word 'path' to mean a 'walk'. (See for example, [20] and [26].) A graph is said to be *connected* if for any $x, y \in V$, there is a path from x to y. The number of walks from x to y of length r is clearly given by the x, y-th entry of A^r, where again we have adopted the convention of counting a loop with multiplicity 2. A graph is called *k-regular* if every vertex has degree k.

The following theorem is basic in graph theory.

Theorem 1. *Let A be the adjacency matrix of an undirected graph X. Let $\Delta(X)$ be the maximal degree of any vertex of X. If λ is an eigenvalue of A, then $|\lambda| \le \Delta(X)$.*

Proof. Let v be an eigenvector of A corresponding to an eigenvalue λ. Then, $Av = \lambda v$. Write $v = (x_1, ..., x_n)^t$ and assume without loss of generality that $|x_1| = \max_{1 \le i \le n} |x_i|$. Then,

$$|\lambda||x_1| = \left|\sum_{j=1}^{n} a_{1j} x_j\right| \le |x_1| \sum_{j=1}^{n} a_{1j} = |x_1| \deg(v_1) \le |x_1| \Delta(X)$$

from which we deduce $|\lambda| \le \Delta(X)$. This completes the proof. □

Corollary 2. *If X is a k-regular graph, then all the eigenvalues λ of its adjacency matrix satisfy $|\lambda| \le k$.*

Since a k-regular graph is one whose adjacency matrix has every row sum (and hence every column sum) equal to k, we clearly have that $\lambda_0 = k$ is an eigenvalue of A with eigenvector equal to $u = (1, 1, ..., 1)^t$. The following theorem makes this more precise.

Theorem 3. *If X is a k-regular graph, then $\lambda = k$ is an eigenvalue with multiplicity equal to the number of connected components of X.*

Proof. We have already proved the first part. Let $v = (x_1, ..., x_n)^t$ be an eigenvector of A with eigenvalue k. Without loss of generality, suppose $|x_1| = \max_{1 \le i \le n} |x_i|$. We may also suppose $x_1 > 0$. Then

$$kx_1 = \sum_{j=1}^{n} a_{1j} x_j \le \sum_{j=1}^{n} a_{1j} x_1 = kx_1.$$

This means, that every j for which $a_{1j} \neq 0$, we must have $x_j = x_1$. In particular, this is true for all j for which v_j is adjacent to v_1. Repeating the argument with each of the neighbouring vertices, we deduce that $x_j = x_1$ if v_j is connected to v_1. As we may duplicate this argument for each component, the result is now clear. $\square$

Thus, if X is a connected k-regular graph, we may arrange the eigenvalues as

$$k = \lambda_0(X) > \lambda_1(X) \ge \cdots \ge \lambda_{n-1}(X) \ge -k.$$

It is not difficult to show that $-k$ is an eigenvalue of X if and only if X is bipartite, in which case its multiplicity is again equal to the number of connected components. Any eigenvalue $\lambda_i \neq \pm k$ is referred to as a non-trivial eigenvalue. We denote by $\lambda(X)$ the maximum of the absolute values of all the non-trivial eigenvalues. A *Ramanujan multigraph* is a k-regular graph satisfying

$$\lambda(X) \le 2\sqrt{k-1}.$$

A *Ramanujan graph* is a Ramanujan multigraph having no multiple edges or loops. The motivation for these definitions will become apparent in later sections. The significance of such graphs will also be elaborated upon later. For the moment, let us state that the explicit construction (see section 5 below) of such graphs for a fixed k and $n \to \infty$ has only been described in the case $k-1$ is prime [12], [14] or a prime power [15] and it is still an open problem in the general case. Thus, the simplest case that is open is when $k = 7$. That is, we must construct a family of 7-regular graphs X_i with $|X_i|$ tending to infinity whose corresponding adjacency matrices have non-trivial eigenvalues λ satisfying $|\lambda| \le 2\sqrt{6}$. In this context, Pizer [18] constructs what he calls 'almost' Ramanujan graphs by using the theory of Hecke operators. More precisely, he shows that for every k, there is a family of k-regular graphs X_i with $|X_i|$ tending to infinity and the non-trivial eigenvalues λ of the corresponding adjacency matrices satisfy the inequality $|\lambda| \leq d(k-1)\sqrt{k-1}$ where $d(k-1)$ denotes the number of positive divisors of $k-1$.

The complete graph K_r as well as the bipartite graph $K_{r,r}$ are easily seen to be Ramanujan (see the discussion in section 2). The Petersen graph (see Figure 1) is a 3-regular graph whose adjacency matrix has characteristic polynomial $(\lambda-3)(\lambda+2)^4(\lambda-1)^5$, and thus is easily seen to be Ramanujan.

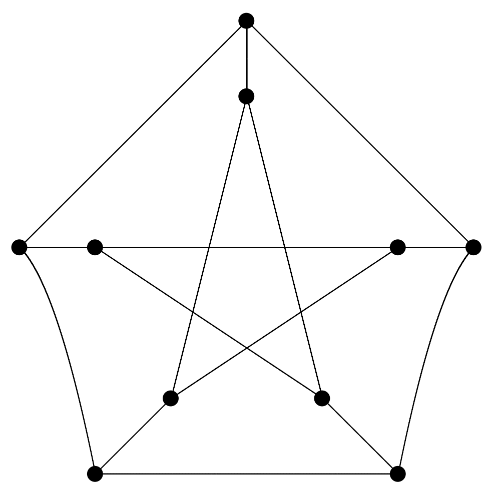

Figure 1. The Petersen Graph

Friedman [5] has shown that random k-regular graphs are close to being Ramanujan in the sense that λ_1 (as defined above) satisfies

$$\lambda_1 \leq 2\sqrt{k-1} + 2\log k + O(1).$$

1. Preliminaries

We can define a metric on a connected graph by defining the distance $d(x, y)$ for $x, y \in V$ as the minimal length amongst all the paths from x to y. The *diameter* of a connected graph is then the maximum value of the distance function. We begin by deriving a simple estimate for the diameter of a k-regular graph involving $\lambda(X)$ due to Chung [3]. If A is the adjacency matrix, then the (x, y)-th entry of A^r is the number of walks from x to y of length r. Hence, if every entry of A^m is strictly positive, then the diameter of X is at most m. We will use this observation below to derive an upper bound for the diameter of a k-regular graph.

Let $n = |V|$ and $u_0, u_1, ..., u_{n-1}$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\lambda_0, ..., \lambda_{n-1}$ respectively. We may take $u_0 = u/\sqrt{n}$ where $u = (1, 1, ..., 1)^t$ as defined earlier. We can write

$$A = \sum_{i=0}^{n-1} \lambda_i u_i u_i^t.$$

More generally,

$$A^r = \sum_{i=0}^{n-1} \lambda_i^r u_i u_i^t.$$

In particular, we see that the (x, y)-th entry of A^m is

$$= \sum_i \lambda_i^m (u_i u_i^t)_{x,y}.$$

If X is a connected k-regular graph, $\lambda_0 = k$ and the above expression is

$$\geq \frac{k^m}{n} - \Big| \sum_{i \geq 1} \lambda_i^m (u_i)_x (u_i)_y \Big|.$$

Let us assume that X is not bipartite (so that $-k$ is not an eigenvalue). Then, by the Cauchy-Schwarz inequality,

$$\Big| \sum_{i \geq 1} \lambda_i^m (u_i)_x (u_i)_y \Big| \leq \lambda(X)^m \left(\sum_{i \geq 1} (u_i)_x^2 \right)^{1/2} \left(\sum_{i \geq 1} (u_i)_y^2 \right)^{1/2}.$$

Recalling that the u_i's form an orthonormal basis, this is easily seen to be

$$\leq \lambda(X)^m (1 - (u_0)_x^2)^{1/2} (1 - (u_0)_y^2)^{1/2} \leq \lambda(X)^m (1 - 1/n).$$

Thus, the (x, y)-th entry of A^m is always positive if

$$\frac{k^m}{\lambda(X)^m} > n - 1.$$

In other words, we have proved

Theorem 4. *(Chung, 1989) Let X be a k-regular graph with n vertices and diameter m. If X is not bipartite, then*

$$m \leq \frac{\log(n-1)}{\log(k/\lambda(X))} + 1.$$

A similar result can be derived for k-regular bipartite graphs. In fact, by a minor modification of the above proof, one can show that for bipartite k-regular graphs, we have (see [19])

$$m \leq \frac{\log(n-2)/2}{\log(k/\lambda(X))} + 2.$$

The inequality of Theorem 4 shows that the diameter is minimized by minimizing $\lambda(X)$. (Theorem 5 below also shows this.) In communication theory, one requires the network to have small diameter for efficient operation.

There is another elementary observation about the eigenvalue $\lambda(X)$ that is worth making. Observe that the eigenvalues of AA^t are simply the squares of the eigenvalues of A. On the other hand, the trace of AA^t is simply kn for a k-regular graph X. Thus, if X is not bipartite,

$$k^2 + (n-1)\lambda(X)^2 \geq kn$$

which gives the inequality

$$\lambda(X) \geq \left(\frac{n-k}{n-1}\right)^{1/2} \sqrt{k}.$$

If X is bipartite, then

$$2k^2 + (n-2)\lambda(X)^2 \geq nk,$$

in which case

$$\lambda(X) \geq \left(\frac{n-2k}{n-2}\right)^{1/2} \sqrt{k}.$$

If we think of k as fixed and $n \to \infty$, then we see that

$$\lim_{n \to \infty} \lambda(X) \geq \sqrt{k}.$$

A theorem due to Alon and Boppana (see [12]) asserts that

$$\liminf_{n \to \infty} \lambda(X_{n,k}) \geq 2\sqrt{k-1}$$

where the lim inf is taken over k regular graphs with n going to infinity. Several proofs of this result exist in the literature [12], [9]. A sharper version was derived by Nilli [17] (who is also known as N. Alon):

Theorem 5. *(Nilli, 1991) Suppose that X is a k-regular graph. Assume that the diameter of X is $\geq 2b+2 \geq 4$. Then*

$$\lambda_1(X) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}.$$

To keep this paper self-contained, we give the proof in section 3 below.

Let us make the following observation. If $m = d(u, v)$ is the diameter of X, then the number of walks from u of length m is $\leq k^m$ and as each such walk has $m + 1$ vertices, we deduce that the number of vertices n satisfies the inequality

$$n \leq (m + 1)k^m.$$

Thus, if k is fixed and $n \to \infty$, then the diameter also tends to infinity. In particular, Theorem 5 implies the Alon-Boppana theorem since $\lambda(X) \geq \lambda_1(X)$.

2. Cayley graphs

There is a simple procedure for constructing k-regular graphs using group theory. This can be described as follows. Let G be a finite group and S a k-element multiset of G. That is, S has k elements where we allow repetitions. We suppose that S is *symmetric* in the sense that $s \in S$ implies $s^{-1} \in S$ (with the same multiplicity). Now construct the graph $X(G, S)$ by having the vertex set to be the elements of G with (x, y) an edge if and only if $x^{-1}y \in S$. Since S is allowed to be a multiset, $X(G, S)$ may have multiple edges.

If G is abelian, the eigenvalues of the Cayley graph are easily determined as follows. The cognoscentii will recognize that it is the classical calculation of the Dedekind determinant in number theory.

Theorem 6. *Let G be a finite abelian group and S a symmetric subset of G of size k. Then the eigenvalues of the adjacency matrix of $X(G, S)$ are given by*

$$\lambda_\chi = \sum_{s \in S} \chi(s)$$

as χ ranges over all the irreducible characters of G.

Remark. Notice that for the trivial character, we have $\lambda_0 = k$. If we have for all $\chi \neq 1$

$$\left| \sum_{s \in S} \chi(s) \right| < k$$

then the graph is connected by our earlier remarks. Thus, to construct Ramanujan graphs, we require

$$\left| \sum_{s \in S} \chi(s) \right| \leq 2\sqrt{k - 1}$$

for every non-trivial irreducible character χ of G. This is the strategy employed in many of the explicit constructions of Ramanujan graphs.

Proof. For each irreducible character χ, let v_χ denote the vector $(\chi(g) : g \in G)$. Let $\delta_S(g)$ equal 0 if $g \notin S$, and m if $g \in S$ with multiplicity m. Denote by A the adjacency matrix of $X(G, S)$. Thus, the i, j-th entry of A is the number of edges between i and j. Then,

$$(Av_\chi)_x = \sum_{g\in G} \delta_S(x^{-1}g)\chi(g).$$

By replacing $x^{-1}g$ by s, we obtain

$$(Av_\chi)_x = \chi(x)\left(\sum_{s\in S}\chi(s)\right)$$

which shows that v_χ is an eigenvector with eigenvalue

$$\sum_{s\in S}\chi(s)$$

which completes the proof. □

We remark that in the above proof, we did not really use the symmetry of the set S and so the result extends to Cayley digraphs as well.

As mentioned above, this calculation is reminiscent of the Dedekind determinant formula in number theory. Recall that this formula computes det A where A is the matrix whose (i, j)-th entry is $f(ij^{-1})$ for any function f defined on the finite abelian group G of order n. The determinant is

$$\prod_\chi \left(\sum_{g\in G} f(g)\chi(g)\right).$$

The proof is analogous to the calculation in the proof of Theorem 6 and we leave it to the reader. As an application, it allows us to compute the determinant of a circulant matrix. For instance, we can compute the characteristic polynomial of the complete graph. Indeed, it is not hard to see that by taking the additive cyclic group of order n and setting $f(0) = -\lambda$, $f(a) = 1$ for $a \neq 0$, we obtain that the characteristic polynomial is

$$(\lambda - (n-1))(\lambda + 1)^{n-1}$$

by the Dedekind determinant formula. As the complete graph of order n is an $(n-1)$-regular graph, we see immediately from the above calculation that it is a Ramanujan graph.

Another example of a Ramanujan graph is the bipartite graph $K_{r,r}$. This is an r-regular graph whose adjacency matrix has eigenvalues equal to r, $-r$ and 0 as is easily checked. In fact, its characteristic polynomial is

$$(\lambda - r)(\lambda + r)\lambda^{2r-2}.$$

If G is an abelian group and S is a subset of G, we can define another set of graphs $Y(G, S)$ called *sum graphs* as follows. The vertices consist of elements of G and (x, y) is an edge if $xy \in S$. If we allow S to be a multiset, then we get a graph with multiple edges. Arguing as before, we can show (see [[9], p. 197]):

Theorem 7. *Let G be an abelian group. For each character χ of G, the eigenvalues of $Y(G, S)$ are given as follows. Define*

$$e_\chi = \sum_{s \in S} \chi(s).$$

If $e_\chi = 0$, then v_χ and $v_{\chi^{-1}}$ are both eigenvectors with eigenvalues zero. If $e_\chi \neq 0$, then

$$|e_\chi| v_\chi \pm e_\chi v_{\chi^{-1}}$$

are two eigenvectors with eigenvalues $\pm|e_\chi|$.

To begin, a simple example can be given using Gauss sums. For p an odd prime, let $G = \mathbb{Z}/p\mathbb{Z}$ and S be the multiset of squares. The multigraph $X(G, S)$ is easily seen to be Ramanujan in view of the fact (see for example, [[16], p. 81]):

$$\left| \sum_{x \in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i a x^2/p} \right| = \sqrt{p}$$

for any $a \neq 0$. By our convention in the computation of degree of a vertex, we see that $X(G, S)$ is a $p+1$-regular graph.

Using Theorem 7, Winnie Li [10] constructed Ramanujan graphs in the following way. Let $\mathbb{F}_q$ denote the finite field of q elements. Let $G = \mathbb{F}_{q^2}$ and take for S the elements of G of norm 1. This is a symmetric subset of G and the Cayley graph $X(G, S)$ turns out to be Ramanujan. The latter is a consequence of a theorem of Weil estimating Kloosterman sums (see [22]).

These results allow us to construct Ramanujan graphs by estimating character sums. However, by allowing S to be a multiset, the construction of Ramanujan multigraphs is slightly simplified, as the following theorem shows.

Theorem 8. *Let $G = \mathbb{F}_q$ be a finite field of $q = p^m$ elements and $f(x)$ a polynomial with coefficients in $\mathbb{F}_q$ and of degree 2 or 3. Let S be the multiset*

$$\{f(x) : x \in \mathbb{F}_q\}.$$

Suppose S is symmetric. Then, $Y(G, S)$ is a Ramanujan graph.

The required character sum estimates come from Weil's proof of the Riemann hypothesis for the zeta functions of curves over finite fields. In particular, we have for all $a \in \mathbb{F}_q$, $a \neq 0$,

$$\left| \sum_{x \in \mathbb{F}_q} \exp(2\pi i \operatorname{tr}_{\mathbb{F}_q/\mathbb{F}_p}(af(x))/p) \right| \leq (\deg f - 1)\sqrt{q}$$

provided f is not identically zero (see [[9], p. 94]). In particular, if f has degree 3, we get the estimate of $2\sqrt{q}$ for the exponential sum. For example, if $u \in \mathbb{Z}/p\mathbb{Z}$ and we take

$$S = \{x^3 + ux : x \in \mathbb{Z}/p\mathbb{Z}\},$$

then S is symmetric and, according to our convention, $X(G, S)$ is a k-regular graph with $k = p + 1$. In addition, it is a Ramanujan graph since

$$\left| \sum_{x \in \mathbb{Z}/p\mathbb{Z}} \exp(2\pi i a(x^3 + ux)/p) \right| \leq 2\sqrt{p}$$

by virtue of the Riemann hypothesis for curves (proved by Weil).

There is a generalization of these results to the non-abelian context. This is essentially contained in a paper by Diaconis and Shahshahani [4]. Using their results, one can easily generalize the Dedekind determinant formula as follows (and which does not seem to be widely known). Let G be a finite group and f a class function on G. Then the determinant of the matrix A whose rows (and columns) are indexed by the elements of G and whose (i, j)-th entry is $f(ij^{-1})$ is given by

$$\prod_{\chi} \left(\frac{1}{\chi(1)} \sum_{g \in G} f(g)\chi(g) \right)^{\chi(1)}$$

with the product over the distinct irreducible characters of G.

The following theorem is implicitly contained in [[4], p. 175].

Theorem 9. *Let G be a finite group and S a symmetric subset which is stable under conjugation. Let A be the adjacency matrix of the graph $X(G, S)$ (where $u, v \in G$ are adjacent if and only if $uv^{-1} \in S$). Then the eigenvalues of A are given by*

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$$

as χ ranges over all irreducible characters of G. Moroever, the multiplicity of λ_χ is $\chi(1)^2$.

We remark that the λ_χ in the above theorem need not be all distinct. For example, if there is a non-trivial character χ which is trivial on S, then the multiplicity of the eigenvalue $|S|$ is at least $1 + \chi(1)^2$.

Proof. We essentially modify the proof on pp. 176–177 of [4] to suit our context. We consider the group algebra $\mathbb{C}[G]$ with basis vectors e_g with $g \in G$ and multiplication defined as usual by $e_g e_h = e_{gh}$. We define Q to be the linear operator that acts on $\mathbb{C}[G]$ by left multiplication by

$$\sum_{s \in S} e_s = \sum_{g \in G} \delta_S(g) e_g.$$

The matrix representation of Q with respect to the basis vectors e_g with $g \in G$ is precisely the adjacency matrix of $X(G, S)$ as is easily checked. If r denotes the left regular representation of G on $\mathbb{C}[G]$, we find that the action of

$$r(A) = \sum_{s \in S} r(s)$$

on $\mathbb{C}[G]$ is identical to Q. Moreover, $\mathbb{C}[G]$ decomposes as

$$\mathbb{C}[G] = \oplus_\rho V_\rho$$

where the direct sum is over non-equivalent irreducible representations of G and the subspace V_ρ is a direct sum of $\deg\rho$ copies of the subspace W_ρ corresponding to the irreducible representation ρ. The result is now clear from basic facts of linear algebra. □

We refer the reader to [1] for a more detailed proof of the above in a slightly general context.

3. Relating the diameter and λ_1

In this section, we will give the promised proof of the following result of Nilli [17]:

Theorem 5. *Let X be a k-regular graph. If the diameter of X is $\geq 2b+2 \geq 4$, then*

$$\lambda_1(X) > 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}.$$

As we remarked earlier, the diameter goes to infinity as $|X|$ goes to infinity. Thus, we deduce:

Theorem 10. *(Alon-Boppana)*

$$\liminf_{n\to\infty} \lambda(X_{n,k}) \geq 2\sqrt{k-1}$$

where $X_{n,k}$ denotes a k-regular graph with n vertices.

We preface our proof of Theorem 2 with a few remarks from linear algebra. Let A be a symmetric matrix (a similar analysis applies to Hermitian matrices). Let $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and smallest eigenvalues of A respectively. Then, (see [[7], p. 176]) we have

$$\lambda_{\max} = \max_{v\neq 0} \frac{(Av, v)}{(v, v)}$$

and

$$\lambda_{\min} = \min_{v\neq 0} \frac{(Av, v)}{(v, v)}.$$

To see this, observe that if U denotes the matrix whose columns form an orthonormal basis of eigenvectors of A, then we may write

$$A = UDU^t$$

where D is a diagonal matrix whose diagonal entries are the eigenvalues of A. Thus,

$$(Av, v) = v^tAv = v^tUDU^tv = \sum_i \lambda_i |(U^tv)_i|^2.$$

As each of the terms $|(U^tv)_i|^2$ is non-negative,

$$\lambda_{\min} \sum_i |(U^tv)_i|^2 \leq v^tAv \leq \lambda_{\max} \sum_i |(U^tv)_i|^2.$$

Since U is an orthogonal matrix, we have

$$\sum_I |(U^tv)_i|^2 = \sum_i |v_i|^2 = v^tv.$$

Thus, if $v \neq 0$,

$$\lambda_{\min} \leq \frac{(Av, v)}{(v, v)} \leq \lambda_{\max}.$$

The inequalities are easily seen to be sharp by considering the eigenvectors corresponding to $\lambda_{\max}$ and $\lambda_{\min}$ respectively, which proves our assertion. This result is usually referred to as the Rayleigh-Ritz theorem in the literature.

Now let $L(X)$ denote the space of real-valued functions on X. We can equip the vector space $L(X)$ with an inner product by defining

$$(f, g) = \sum_{x \in X} f(x)g(x).$$

We can view the adjacency matrix as acting on $L(X)$ via the formula

$$(Af)(x) = \sum_{(x,y) \in E} f(y).$$

For a connected k-regular graph, $\lambda_0 = k$ is an eigenvalue of multiplicity 1 and the corresponding eigenspace is the set of constant functions. Hence, we can decompose our space as

$$L(X) = \mathbb{R} f_0 \oplus L_0(X)$$

where $f_0 \equiv 1$ and $L_0(X)$ is the space of functions orthogonal to f_0. Thus, we can consider A as operating on $L_0(X)$. By the Rayleigh-Ritz theorem,

$$\lambda_1(X) = \max_{\substack{f \neq 0 \\ (f, f_0) = 0}} \frac{(Af, f)}{(f, f)}.$$

Since we want a lower bound for $\lambda_1(X)$, it is natural to consider the matrix $\Delta = kI - A$ whose eigenvalues are easily seen to be $k - \lambda_i$ $(0 \leq i \leq n-1)$. (Δ is a discrete analogue of the classical Laplace operator.) Thus,

$$k - \lambda_1(X) = \min_{\substack{f \neq 0 \\ (f, f_0) = 0}} \frac{(\Delta f, f)}{(f, f)}.$$

The strategy now is to find a function f with $(f, f_0) = 0$, that gives a good upper bound on the quotient. We can now prove Theorem 5. We follow [9].

Proof of Theorem 5. Let $u, v \in G$ be such that $d(u, v) \geq 2b + 2$. For $i \geq 0$, define sets

$$U_i = \{x \in G : d(x, u) = i\}$$
$$V_i = \{x \in G : d(x, v) = i\}.$$

Then, the sets $U_0, U_1, \ldots, U_b, V_0, V_1, \ldots, V_b$ are disjoint, for otherwise, by the triangle inequality we get $d(u, v) \leq 2b$ which is a contradiction. Moreover, no vertex of

$$U = \cup_{i=0}^{b} U_i$$

is adjacent to a vertex in

$$V = \cup_{i=0}^{b} V_i$$

for otherwise $d(u, v) \le 2b+1$ which is again a contradiction. For each vertex in U_i, at least one lies in U_{i-1} and at most $q = k-1$ lie in U_{i+1} (for $i \ge 1$). Thus,

$$|U_{i+1}| \le q|U_i|.$$

By the same logic, $|V_{i+1}| \le q|V_i|$. By induction, we see that $|U_b| \le q^{(b-i)}|U_i|$ and $|V_b| \le q^{(b-i)}|V_i|$ for $i \ge 1$. We will set $f(x) = F_i$ for $x \in U_i$, $f(x) = G_i$ for $x \in V_i$ and zero otherwise, with the F_i and G_i to be chosen later. Now,

$$(f, f) = A_1 + B_1$$

where

$$A_1 = \sum_{i=0}^{b} F_i^2 |U_i|$$

and

$$B_1 = \sum_{i=0}^{b} G_i^2 |V_i|.$$

We now choose $F_0 = \alpha$, $G_0 = \beta$, $F_i = \alpha q^{-(i-1)/2}$ and $G_i = \beta q^{-(i-1)/2}$ for $i \ge 1$. We choose α and β so that $(f, f_0) = 0$.

Now we derive an upper bound for $(\Delta f, f)$. Note that

$$\frac{1}{2} \sum_{(x,y)\in E} (f(x) - f(y))^2 = k(f, f) - (Af, f) = (\Delta f, f)$$

by an easy calculation. Recall that no vertex of U is adjacent to a vertex of V. Moreover, f is non-zero only on $U \cup V$. Thus, if we let A_U denote the sum

$$\frac{1}{2} \sum_{\substack{(x,y)\in E \\ x \text{ or } y \in U}} (f(x) - f(y))^2$$

and let A_V be defined similarly, then

$$(\Delta f, f) = A_U + A_V.$$

If we partition according to the contribution from each U_i and keep in mind that each $x \in U_i$ has at most $q = k-1$ neighbours in U_{i+1}, we obtain

$$A_U \le \sum_{i=1}^{b-1} |U_i| q \left(q^{-(i-1)/2} - q^{-i/2}\right)^2 \alpha^2 + |U_b| q \cdot q^{-(b-1)} \alpha^2.$$

This is easily computed to be

$$= (\sqrt{q}-1)^2\left(|U_1| + |U_2|q^{-1} + \cdots + |U_{b-1}|q^{-(b-2)} + |U_b|q^{-(b-1)}\right)\alpha^2 \\ + \alpha^2(2\sqrt{q}-1)|U_b|q^{-(b-1)}.$$

Because $q^{-(i-1)}|U_i| \geq q^{-i}|U_{i+1}|$, for $i \geq 1$, we have,

$$A_U \leq (\sqrt{q}-1)^2(A_1 - \alpha^2) + (2\sqrt{q}-1)\frac{A_1 - \alpha^2}{b}$$

which is less than

$$\left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{b}\right)A_1.$$

Similarly,

$$A_V < \left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{b}\right)B_1.$$

Combining these inequalities gives

$$k - \lambda_1(X) \leq \frac{A_U + A_V}{A_1 + B_1} < 1 + q - 2\sqrt{q} + \frac{2\sqrt{q}-1}{b}$$

which proves the theorem, since $k = q + 1$. □

The Alon-Boppana theorem can also be deduced from a result of Serre (as noted in [[9], p. 209]). This says that for any $\epsilon > 0$, there exists a positive constant $c = c(\epsilon, k)$ such that for every k-regular graph X, the number of eigenvalues λ of X with $\lambda > (2-\epsilon)\sqrt{k-1}$ is at least cn where n is the number of vertices of X. Thus, every k-regular graph has a positive proportion of eigenvalues larger than $(2-\epsilon)\sqrt{k-1}$.

4. Expanders

For any subset A of a graph X, we may define the *boundary* of A, denoted ∂A, by

$$\partial A = \{y \in X : d(y, A) = 1\}.$$

That is, the boundary of A consists of vertices which are adjacent to some vertex in A. Let c be a positive real number. A k-regular graph X with n vertices is called a *c-expander* if

$$\frac{|\partial A|}{|A|} \geq c$$

for all subsets A with $|A| \le |X|/2$. Expander graphs play an important role in computer science and the theory of communication networks (see [2]). These graphs arise in questions about designing networks that connect many users while using only a small number of switches. Our interest in them lies in the fact that the theory of c-expanders can be related to the eigenvalue questions of the previous section.

The idea is to apply the Rayleigh-Ritz ratio in the following way. As observed in the previous section, let f be a function orthogonal to the constant function. Then

$$\frac{(\Delta f, f)}{(f, f)} \ge k - \lambda_1(X)$$

for a k-regular graph. Fix a subset A of X. If we set

$$f(x) = \begin{cases} |X \backslash A| & \text{if } x \in A \\ -|A| & \text{if } x \notin A \end{cases}$$

then it is easily seen that $(f, f_0) = 0$. On the other hand, a direct calculation shows that

$$(f, f) = |X||A||X \backslash A|.$$

By using the formula

$$(\Delta f, f) = \frac{1}{2} \sum_{(x,y) \in E} (f(x) - f(y))^2$$

we easily check that

$$(\Delta f, f) = |X|^2 |\partial A|$$

so that by the Rayleigh-Ritz theorem we obtain

$$\frac{|\partial A|}{|A|} \ge (k - \lambda_1(X)) \frac{|X \backslash A|}{|X|}.$$

By the definition of an expander, we consider only subsets A with $|A| \le |X|/2$, so that $(k - \lambda_1)/2$ is an expander constant for X. Thus, making λ_1 as small as possible gives us good expander graphs. By the Alon-Boppana theorem, we cannot do better than

$$\lambda_1(X) \le \lambda(X) \le 2\sqrt{k-1}.$$

Hence, Ramanujan graphs also make good expanders.

5. Explicit Ramanujan graphs

In this section, we give a brief outline of the explicit construction of Ramanujan graphs due to Lubotzky, Phillips and Sarnak [12]. Let p and q be unequal primes $p, q \equiv 1(\bmod 4)$. Let u be an integer so that $u^2 \equiv -1(\bmod q)$. By a classical formula of Jacobi, we know that there are $8(p+1)$ solutions $v = (a, b, c, d)$ such that $a^2 + b^2 + c^2 + d^2 = p$. Among these, there are exactly $p+1$ with $a > 0$ and b, c, d even, as is easily shown. To each such v we associate the matrix

$$\tilde{v} = \begin{pmatrix} a + ub & c + ud \\ -c + ud & a - ub \end{pmatrix}$$

which gives us $p+1$ matrices in $PGL_2(\mathbb{Z}/q\mathbb{Z})$. We let S be the set of these matrices $\tilde{v}$ and take $G = PGL_2(\mathbb{Z}/q\mathbb{Z})$. In [LPS], it is shown that the Cayley graphs $X(G, S)$ are Ramanujan graphs. As we vary q, we get an infinite family of such graphs, all $p+1$-regular.

6. Counting walks in regular graphs

If A is the adjacency matrix of X, it is clear that the (x, y)-th co-ordinate of A^r enumerates the number of walks of length r from x to y. We will be interested in *proper walks*, that is, walks which do not have back-tracking. We are interested in counting the number of proper walks of length r in a k-regular graph. Let A_r denote the matrix whose (x, y)-th entry will be the number of proper walks from x to y. Then, $A_0 = I$ and $A_1 = A$ and clearly

$$A^2 = A_2 + kI$$

since A_2 encodes the number of proper walks of length 2. Inductively, it is clear that

$$A_1 A_r = A_{r+1} + (k-1)A_{r-1},$$

since the left hand side enumerates walks of length $r+1$ which are extended from proper walks of length r and the right side enumerates first the proper walks of length $r+1$ and proper walks of length $r-1$ which are extended to ‘improper’ walks of length r.

This recursion allows us to deduce the following identity of formal power series:

Proposition 11.

$$\left(\sum_{r=0}^{\infty} A_r t^r\right)\left(I - At + (k-1)t^2 I\right) = (1 - t^2)I.$$

From this result, it is possible to establish the rationality of the zeta function of a regular graph (see Theorem 12 below).

7. The Ihara zeta function

Let X be a k-regular graph and set $q = k - 1$. Motivated by the theory of the Selberg zeta function, Ihara [8] was led to make the following definitions and construct the graph-theoretic analogue of it as follows. A proper walk whose endpoints are equal is called a *closed geodesic*. If γ is a closed geodesic, we denote by γ^r the closed geodesic obtained by repeating the walk γ r times. A closed geodesic which is not the power of another one is called a *prime geodesic*. We define an equivalence relation on the closed geodesics as follows. $(x_0, ..., x_n)$ and $(y_0, ..., y_m)$ are equivalent if and only if $m = n$ and there is a d such that $y_i = x_{i+d}$ for all i (and the subscripts are interpreted modulo n). An equivalence class of a prime geodesic is called a *prime geodesic cycle*. Ihara [Ih] then defines the zeta function

$$Z_X(s) = \prod_p \left(1 - q^{-s\ell(p)}\right)^{-1}$$

where the product is over all prime geodesic cycles p and $\ell(p)$ is the length of p.

Ihara proves the following theorem:

Theorem 12. *For $g = (q-1)|X|/2$, we have*

$$Z_X(s) = (1-u^2)^{-g} \det(I - Au + qu^2 I)^{-1}, \quad u = q^{-s}.$$

Moreover, $Z_X(s)$ satisfies the "Riemann hypothesis" (that is, all the singular points in the region $0 < \Re(s) < 1$ lie on $\Re(s) = 1/2$) if and only if X is a Ramanujan graph.

Proof. (Sketch) We assume that the zeta function has the shape given (see [25]) and show that it satisfies the Riemann hypothesis if and only if X is Ramanujan. Let $\phi(z) = \det(zI - A)$ be the characteristic polynomial of A. If we set $z = (1 + qu^2)/u$, then the singular points of $Z_X(s)$ arise from the zeros of $\phi(z)$. First suppose that $Z_X(s)$ satisfies the "Riemann hypothesis." Then, for any singular point s_0, we have $q|u_0|^2 = 1$ where $u_0 = q^{-s_0}$. Let $z_0 = (1 + qu_0^2)/u_0$ be the corresponding eigenvalue of A. Since,

$$\frac{z\overline{u}}{\overline{u}} = \frac{(1+qu^2)\overline{u}}{u\overline{u}} = \frac{\overline{u} + q|u|^2 u}{|u|^2}$$

we see that

$$|z_0| = q|u_0 + \overline{u_0}| \le 2\sqrt{q}$$

so that X is Ramanujan. Conversely, if X is Ramanujan, $|z_0| \le 2\sqrt{q}$ for any eigenvalue z_0 of A. As z_0 is real, this means $z_0^2 \le 4q$ and

$$u_0 := \frac{z_0 \pm \sqrt{z_0^2 - 4q}}{2q}$$

either is not real or $u_0 = \pm 1/\sqrt{q}$. If the latter is the case, we are done. In the former case, we have as before

$$z_0 = \frac{\overline{u_0} + q|u_0|^2 u_0}{|u_0|^2}$$

and the reality of z_0 forces $q|u_0|^2 = 1$, as desired. $\square$

Hashimoto [6], as well as Stark and Terras [23] have defined a zeta function for an arbitrary graph and established its rationality. The definition of this zeta function is simple enough. Let N_r be the number of closed walks γ of length r so that neither γ nor γ^2 have backtracking. Then, the *zeta function* of the graph X is defined as

$$\mathcal{Z}_X(t) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r t^r}{r}\right).$$

This definition is very similar to the zeta function of an algebraic variety. It would be interesting to interpret the singularities of $\mathcal{Z}_X(t)$ in terms of properties of the graph. For instance, these zeta functions have a pole at $t = 1$ and Hashimoto [6] has shown that the residue at $t = 1$ is related to the number of spanning trees of the graph X. Thus, this number is the graph-theoretic analogue of the class number of an algebraic number field. These constructions raise the intriguing question of whether there is a generalization of the notion of a graph to that of a 'supergraph' whose zeta function would (in some cases) coincide with those higher dimensional zeta functions of varieties.

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