Question 1

First of all, we will be using the following theorem proven in class about the recurrence relation of $|s(n,k)| = S^*(n,k)$.

**Theorem 1.**

$$S^*(n + 1, k) = S^*(n, k - 1) + nS^*(n, k)$$

holds for any $n,k$, and defines a recurrence relation for $S^*(n,k)$.

Let us define $(t)^n = t(t + 1)(t + 2)\cdots(t + n - 1)$ to be the polynomial in $t$ with roots $0, -1, -2, \ldots, -n - 1$. Let us now define $a(n,k)$ as follows

$$(t)^n = \sum_{k=0}^{n} a(n,k)t^k. \quad (1)$$

We want to find a recurrence relation for $a(n,k)$. We have that

$$(t)^{n+1} = (t)^n(t + n)$$

$$= t(t)^n + n(t)^n$$

and so by using equation (1)

$$\sum_{k=0}^{n+1} a(n+1,k)t^k = \sum_{k=0}^{n} a(n,k)t^{k+1} + \sum_{k=0}^{n} na(n,k)t^k$$

$$= \sum_{k=0}^{n+1} a(n,k-1)t^k + \sum_{k=0}^{n} na(n,k)t^k.$$

Thus by equating the coefficient of $t^k$ from both sides, we have

$$a(n+1,k) = a(n,k - 1) + na(n,k).$$

We notice that $a(n,k)$ follows exactly the same recurrence relation as $S^*(n,k)$. If the initial conditions are the same, then it follows that $a(n,k) = S^*(n,k)$.

The coefficient for the term $t$ in $(t)^n$ is simply the constant term of $\frac{(n)^n}{t}$, which is clearly $(n - 1)!$. We thus have $a(n,1) = (n - 1)!$. 

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On the other hand, $S^*(n, 1)$ by definition is the number of permutations of $S_n$ which can be written as a single cycle. There is $n!$ ways to list $n$ elements in a cycle, but because cycling them yields the same permutation, I need to divide this by $n$ to have the number of unique permutation. Therefore, $S^*(n, 1) = (n - 1)!$.

It is also clear that $a(n, 0) = S^*(n, 0) = 0$. Therefore, the initial conditions match and so $a(n, k) = S^*(n, k)$. In particular, replacing in equation (1) yields

$$t(t + 1) \cdots (x + n - 1) = \sum_{k=0}^{n} S^*(n, k) t^k.$$ 

Since it is clear by definition that $|s(n, k)| = S^*(n, k)$, we have the desired result.

**Question 2**

Using the principle of inclusion exclusion and the observation that $k! S(n, k)$ is the number of surjective maps $f : \{n\} \rightarrow \{k\}$, we proved the following theorem.

**Theorem 2.**

$$k! S(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n.$$

In particular, letting $k = 2$, we find that

$$2! S(n, 2) = \sum_{j=0}^{2} (-1)^j \binom{2}{j} (2 - j)^n$$

$$= \binom{2}{0} 2^n - \binom{2}{1} 1^n$$

$$= 2^n - 2$$

and therefore,

$$S(n, 2) = 2^{n-1} - 1.$$ 

**Question 3**

Consider a stack of $m + n$ cards, $n$ of which are black and $m$ of which are red. We will try to find an expression of the number of subsets of $k$ cards consisting only of red cards.

Firstly, it is quite simple to see that this number simply is $\binom{m}{k}$ as we choose $k$ cards amongst the $m$ red cards.
On the other hand, it is possible to use the principle of inclusion exclusion to find another expression for this quantity. Let $A$ be the set of all subsets of $[n + m]$ (i.e. of cards) of size $k$. Clearly, $|A| = \binom{m+n}{k} = N$.

For every $i \in [N]$, we define $A_i \subseteq A$ as the $k$-elements subsets of cards for which the $i^{th}$ card is black. Also, if $i > k$ we take $A_i = \emptyset$.

Then, the number of $k$-elements subsets of cards only containing red cards is given by

$$\left| A \setminus \bigcup_{i=1}^{N} A_i \right|.$$ 

If for $I \subseteq [N]$ we define $A_I = \cap_{i \in I} A_i$, we know by the principle of inclusion exclusion that

$$\left| A \setminus \bigcup_{i=1}^{N} A_i \right| = \sum_{I \subseteq [N]} (-1)^{|I|} |A_I|.$$ 

Notice that if $I \not\subseteq [k]$, then $A_I = \emptyset$ and thus there is no contribution in the sum. Therefore, the right hand side of the equation above becomes

$$\sum_{I \subseteq [k]} (-1)^{|I|} |A_I|.$$ 

Also, for these $I$, we have that $A_I$ is exactly the number of $k$-elements subsets of cards with black cards at positions in $I$. When summing over all $I$ of a same cardinality, this becomes simply the number of such subsets with at least $|I|$ black cards. This is given by $\binom{n}{|I|} \binom{m+n-|I|}{k-|I|}$. Therefore, we can once again replace the sum above by indexing it only with possible sizes of $I$. We have that the number of $k$-elements subsets of only red cards is

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} \binom{m+n-i}{k-i}.$$ 

Notice that for all $i$ between $k$ and $n$, the summand is 0 as one of the two binomial coefficients is 0. Therefore, we can freely change the upper bound of the sum to $n$. Finally, as we already had an expression for this number, both have to be equal, that is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+n-i}{k-i} = \binom{m}{k}.$$ 

**Question 4**

Suppose that $X$ is a finite connected graph. Since $X$ is finite, there exists a vertex $v \in V(X)$ such that $\text{rad}(X) = \text{ecc}(v)$, the eccentricity of $v$. Similarly, there exists a vertex $u \in V(X)$ such that $\text{diam}(X) = \text{ecc}(u)$, and moreover, that eccentricity is realised as a distance from $u$. 

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to (at least) one vertex \( w \in V(X) \). Consequently, \( \text{diam}(X) = d(u, w) \). Since the distance on a graph constitutes a metric, we know that the triangle inequality holds, and therefore that

\[
d(u, w) \leq d(u, v) + d(v, w).
\]

But since any distance from \( v \) is at most its eccentricity,

\[
\leq 2 \text{ecc}(v)
\]

\[
= 2 \text{rad}(X).
\]

Therefore, \( \text{diam}(X) \leq 2 \text{rad}(X) \).

**Question 5**

Let \( M \) be the incidence matrix of \( X \), and called the \( ij \)th entry \( m_{ij} \), i.e. the entry associated with vertex \( v_i \) and edge \( e_j \). Recall that \( m_{ij} \) is 1 if edge \( e_j \) is incident to vertex \( v_i \) and 0 otherwise. Then, consider the matrix \( M^tM = (a_{ij})_{1 \leq i,j \leq e} \) where \( e \) is the number of edges in \( X \). Then, using the matrix multiplication rule,

\[
a_{jj} = \sum_{v_i \in V(X)} (m_{ij})^2.
\]

However, the only summands which are nonzero in the above sum are those for the vertices at the endpoints of edge \( e_j \). In particular, for any edge, there are exactly 2 of those, each contributing 1. Therefore, \( a_{jj} = 2 \) for all edges \( e_j \). Therefore,

\[
\text{tr}(M^tM) = \sum_{e_j \in E(X)} 2 = 2e.
\]

**Question 6**

Let \( X \) be a simple graph with \( n \) vertices and \( A_X \) its adjacency matrix. Choose an orientation for \( X \), let \( N \) be the oriented incidence matrix and \( n_{ik} \) its entry corresponding to vertex \( v_i \) and edge \( e_k \). Let \( NN^t = (b_{ij})_{1 \leq i,j \leq n} \), i.e. name \( b_{ij} \) the \( ij \)th entry of \( NN^t \).

For \( i \neq j \), using the multiplication rule for matrices, we see that

\[
b_{ij} = \sum_{e_k \in E(X)} n_{ik}n_{jk}.
\]

However, it is easy to see that \( n_{ik}n_{jk} \) is -1 if edge \( e_k \) is either \((v_i, v_j)\) or \((v_j, v_i)\), and 0 otherwise. Since there can be only a single edge between \( v_i \) and \( v_j \), we conclude that \( b_{ij} \) is -1 if the two vertices are adjacent, and 0 otherwise. Note that this is exactly the negative of the \( ij \)th entry of \( A_X \).
Also, using the same rule,
\[ b_{ii} = \sum_{e_k \in E(X)} n_{ik}^2. \]
However, \( n_{ik}^2 \) is 1 whenever \( e_k \) is incident to \( v_i \), and 0 otherwise. Therefore, summing over all the edges, we find that \( b_{ii} = d(v_i) \).

Let \( D \) corresponds to the matrix with \( d(v_i) \) at the \( i \)th diagonal entry and 0 everywhere else. Since \( AX \) has all its diagonal entries 0, the matrix \( D - AX \) has entry \( d(v_i) \) on the diagonal, and those of \( -AX \) everywhere else, and so is exactly \( NN^t \).

**Question 7**

**Proposition 3.** The polynomial \( \lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 1 \) cannot be the characteristic polynomial of an adjacency matrix of any simple graph.

**Proof.** Suppose that \( X \) is a simple graph with characteristic polynomial \( \chi(\lambda) = \lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 1 \). Then, notice that \( \chi(\lambda) = (\lambda^2 + 1)(\lambda^2 + \lambda + 1) \). Since \( \sqrt{-1} \) is a zero of \( \chi(\lambda) \), it is an eigenvalue of \( AX \). However, as \( X \) is a simple graph, its adjacency matrix \( AX \) is symmetric. Therefore, we know that all its eigenvalues must be real. We thus have a contradiction, and so no such \( X \) can exist. \( \square \)

**Question 8**

From Theorem 2 in question 2, we know that
\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n = k!S(n, k). \]
Therefore, by extension,
\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^{n+1} = k!S(n + 1, k). \]

If we let \( k = n \), we then obtain
\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - j)^{n+1} = n!S(n + 1, n). \]

Note that \( S(n + 1, n) \) represents the number of ways to partition \( n + 1 \) numbers into \( n \) blocks. This amounts to the number of ways of choosing which two numbers are going to share the same block, i.e. \( \binom{n+1}{2} \). Therefore,
\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - j)^{n+1} = n!\binom{n+1}{2}. \]
Question 9

Problem 4. Let \( s(n) \) denotes the number of elements \( \sigma \in S_n \) such that \( \sigma^2 = 1 \). Show that

\[
\sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} = \exp(t + t^2/2).
\]

In order to solve this problem, we first need the following lemma.

Lemma 5. The number of elements of \( S_n \) having a cycle decomposition consisting only of \( j \) transpositions (cycles of length 2) and (possibly) fixed points is

\[
s_j(n) = \frac{n!}{2^j j!(n - 2j)!}
\]

if \( j \leq \frac{n}{2} \) and \( s_j(n) = 0 \) otherwise.

Proof. Clearly if \( j > \frac{n}{2} \), there is not enough distinct elements to construct that many cycles, and so there is no such \( \sigma \in S_n \). Otherwise, the number of such elements is given by choosing successively the pairs of numbers to be transposed, divided by the number of ways to permute these pairs around. As such,

\[
s_j(n) = \frac{1}{j!} \binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2(j-1)}{2}
\]

\[
= \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}.
\]

Expanding the binomial coefficients, we can see a lot of cancellation happening as

\[
= \frac{n!(n-2)! \cdots (n-2(j-1))!}{j! 2^j (n-2)! \cdots (n-2(j-1))! (n-2j)!}
\]

\[
= \frac{n!}{j! 2^j (n-2j)!}.
\]

\[
\square
\]

Solution (Problem 4). First of all, notice that if \( \sigma \in S_n \) is such that \( \sigma^2 = 1 \), then every cycle in the decomposition of \( \sigma \) must have length dividing 2. Indeed, squaring a cycle yields the new cycle given by sending a number to its second neighbor on the right in the original cycle. For all numbers to be sent to itself, all cycles must have length 1 or 2.
It therefore follows that

\[ s(n) = \sum_{j=0}^{n/2} s_j(n) \]

and since \( s_j(n) \) is taken to be 0 if \( j > n/2 \), we can make the upper limit of summation arbitrarily large, i.e.

\[ s(n) = \sum_{j=0}^{\infty} s_j(n). \]

Substituting in the statement of Problem 4, we get

\[ \sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{s_j(n)t^n}{n!}. \]

Since everything involved is positive, the summands are all positive and so we can exchange the order of summation.

\[ = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{s_j(n)t^n}{n!}. \]

Since \( s_j(n) \) is 0 for \( n < 2j \), we have

\[ = \sum_{j=0}^{\infty} \sum_{n=2j}^{\infty} \frac{s_j(n)t^n}{n!} \]

and replacing the value of \( s_j(n) \) for other \( n \geq 2j \) given in Lemma 5, we have

\[ = \sum_{j=0}^{\infty} \sum_{n=2j}^{\infty} \frac{t^{2j}}{2j!} \frac{t^{n-2j}}{(n-2j)!} \]

\[ = \sum_{j=0}^{\infty} \frac{t^{2j}}{2j!} \sum_{n=2j}^{\infty} \frac{t^{n-2j}}{(n-2j)!} \]

Finally, replacing \( k = n - 2j \), we get

\[ = \sum_{j=0}^{\infty} \frac{t^{2j}}{2j!} \sum_{k=0}^{\infty} \frac{t^k}{k!} \]
and we can therefore separate the sums as

\[
\begin{align*}
&= \left( \sum_{j=0}^{\infty} \frac{(t^2/2)^j}{j!} \right) \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \\
&= \exp(t^2/2) \exp(t) \\
&= \exp(t + t^2/2)
\end{align*}
\]

and we have the desired result.

**Question 10**

First, we begin by noting the following corollary of Theorem 2.

**Corollary 6.** For all \( k > n \),

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n = 0.
\]

This is a simple consequence of Theorem 2 as the above sum is simply \( k!S(n,k) \) with \( k > n \). However, since there are no ways of dividing \( n \) numbers in \( k \) blocks, \( S(n,k) = 0 \).

Therefore, we have that

\[
B_n = \sum_{k=0}^{n} S(n,k) \\
= \sum_{k=0}^{\infty} S(n,k)
\]

by Corollary 6,

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \frac{1}{k!}
\]

by Theorem 2, and since \( \binom{k}{j} = 0 \) for \( j > k \), we have

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} (k-j)^n \frac{1}{k!}.
\]
It should be clear by comparing the order of magnitude of \((k - j)^n\) and \(j!(k - j)\!\) (using Stirling’s formula for example) that the sum above converges absolutely. Therefore, we exchange the order of summation to get

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j \binom{k}{j} (k - j)^n \frac{1}{k!}
\]

and as \(\binom{k}{j} = 0\) for \(k < j\), we now have

\[
= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} (-1)^j \binom{k}{j} (k - j)^n \frac{1}{k!}.
\]

Expanding the binomial coefficient and canceling \(k!\), we obtain

\[
= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{(-1)^j (k - j)^n}{j! (k - j)!}.
\]

Finally, setting \(l = k - j\), we get

\[
= \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^j (l)^n}{j! l!}
= \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\right) \left(\sum_{l=0}^{\infty} \frac{(l)^n}{l!}\right)
= \frac{1}{e} \sum_{l=0}^{\infty} \frac{l^n}{l!}.
\]

Therefore, we have that

\[
B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.
\]