MATH 802: ENUMERATIVE COMBINATORICS
ASSIGNMENT 2

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Facts

Recall that, the Stirling number $S(k,n)$ of the second kind is defined as the number of partitions of a $[k]$ into n non-empty blocks. We recall the following result from class:

Lemma 0.1. *The following numbers are equal:*

(1) *The number of surjective maps from $[k]$ to $[n]$.*
(2) *$S(k,n)n!$ where $S(k,n)$ denotes the Stirling number of the second kind.*
(3) $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k$.

Proof. (1) = (2): Given a partition of $[k]$ into n blocks, one may define a surjective map by mapping distinct parts to distinct numbers between 1 and n. There are $n!$ such surjections.

(1) = (3): (Principle of Inclusion and Exclusion.) Let A_i denote the set of maps $f : [k] \to [n]$ such that $i \notin \mathrm{Im}(f)$. Then, it follows that the number of surjective maps $[k] \to [n]$ is given by

$$\left|[n]^{[k]} \setminus \bigcup_{i=1}^{k} A_i\right|.$$

Applying the principle of inclusion-exclusion, we conclude:

$$\begin{aligned}\left|[n]^{[k]} \setminus \bigcup_{i=1}^{k} A_i\right| &= \sum_{I\subseteq[k]}(-1)^{|I|}A_I \\ &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k.\end{aligned}$$

This completes the proof. □

Problems

Problem 1. *Let $S(n,k)$ be the number of ways of partitioning an n-set into k blocks. Prove that $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$ and $S(n,n-1) = \binom{n}{2}$, Find a formula for $S(n,n-2)$.*

Solution.

Proof that $S(n,1) = 1$: There is exactly one partition of an n-set which consists of just one block, viz, $[n]$. So, $S(n,1) = 1$.

Proof that $S(n,2) = 2^{n-1} - 1$: Let $\Sigma(n,2)$ denote the set of all partitions of $[n]$ into 2 parts (so that $|\Sigma(n,2)| = S(n,2)$). Note that any element of $\Sigma(n,2)$ is of the form $A \coprod A^c$ for a subset A of $[n]$ such that $A \neq \varnothing \neq A^c$. Consider the map:

$$\begin{aligned} \phi : 2^{[n]} \setminus \{\varnothing, [n]\} &\to \Sigma(n,k) \\ A &\mapsto A \coprod A^c \end{aligned}$$

Then, this map is clearly surjective and is 2-to-1, indeed $\phi(A) = \phi(A^c)$. Thus,

$$2|\Sigma(n,k)| = |2^{[n]} \setminus \{\varnothing, [n]\}| = 2^n - 2$$

whence we have $S(n,2) = 2^{n-1} - 1$ as required.

Proof that $S(n,n-1) = \binom{n}{2}$: Let $\Sigma(n,n-1)$ denote the set of all partitions of $[n]$ into $n-1$ parts (so that $|\Sigma(n,n-1)| = S(n,n-1)$). Note that any element of $\Sigma(n,n-1)$ is of the form $A_1 \coprod A_2 \coprod \cdots \coprod A_{n-1}$ where A_i is non-empty for every i. For a fixed partition of the above form, let $a_i = |A_i|$. Then, we have that $a_i \geqslant 1$ and

$$\sum_{i=1}^{n-1} a_i = n.$$

If there are atleast two indices, say i, j such that $a_i, a_j \geqslant 2$, then, we have that

$$n = \sum_{i=1}^{n-1} a_i \geqslant 2 + 2 + (n-1) - 2 = n + 1$$

a contradiction. On the other hand, if $a_i = 1$ for all i, then, we get the absurd equation $n = n - 1$. Thus, there is precisely one index i such that $a_i = 2$ and $a_j = 1$ for all $j \neq i$. We may let $i = n - 1$ without loss of generality. (This is an elaboration of the Pigeonhole principle.)

The above consideration shows that an element of $\Sigma(n,n-1)$ is determined by the 2-set A_{n-1}. Conversely, every 2-subset T of $[n]$ determines a partition of $[n]$ into $n-1$ blocks:

$$[n] = \coprod_{a \notin T} \{a\} \coprod T.$$

This shows that $S(n,n-1) = \binom{n}{2}$ (the number of 2-subsets of $[n]$).

Formula for $S(n,n-2)$: In this part, we argue as in the previous part, but somewhat more systematically. Let k_c denote the number of blocks with cardinality c in a given partition of $[n]$ into $n-2$ blocks. Then, we have the equation:

$$(n-2) - \sum_{c \geqslant 2} k_c + \sum_{c \geqslant 2} c k_c = n \implies \sum_{c \geqslant 2} (c-1) k_c = 2.$$

This equation shows that if $k_c > 0$, then $c \leqslant 3$. Further it follows that $k_2 = 2, k_3 = 0$ or $k_2 = 0, k_3 = 1$.

The number of partitions of $[n]$ into $n-2$ blocks consisting of two blocks of size 2 (and other blocks singleton) is $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$. The number of partitions of $[n]$ into $n-2$ blocks consisting of one block of size 3 (and other blocks singleton) is $\binom{n}{3}$.

Based on the above analysis, it follows that

$$S(n, n-2) = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{n(n-1)(n-2)(3n-5)}{24}.$$

This completes this problem. □

Remark 0.2. Another way to approach the above problem would be through recurrence established for Stirling numbers $S(n,k)$ of the second kind. Recall from class that

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Set $a_n = S(n, n-2)$; we know that $a_3 = S(3,1) = 1$. On taking $k = n-2$ in the above recurrence relation, we have

$$a_n = a_{n-1} + (n-2)\binom{n-1}{2}$$

using the previous calculation. Rearranging the above equality and summing from $n = 4$ to k gives us:

$$\begin{aligned} a_k - 1 &= \sum_{n=4}^{k}(a_n - a_{n-1}) \\ &= \sum_{n=4}^{k}\frac{(n-1)(n-2)^2}{2} \end{aligned}$$

Making a change of variable in the above sum:

$$\begin{aligned} &= \sum_{n=2}^{k-2}\frac{n^2(n+1)}{2} \\ &= \frac{1}{2}\sum_{n=2}^{k-2} n^3 + n^2 \end{aligned}$$

Using the known formulae:

$$\begin{aligned} &= \frac{1}{2}\left(\frac{(k-2)^2(k-1)^2}{4} - 1 + \frac{(k-2)(k-1)(2k-3)}{6} - 1\right) \\ a_k &= \frac{(k-1)(k-2)}{4}\left(\frac{(k-2)(k-1)}{2} + \frac{2k-3}{3}\right) \\ &= \frac{k(k-1)(k-2)(3k-5)}{24} \end{aligned}$$

This also proves the expression we computed for $S(n, n-2)$ above.

Problem 2. *Let f_n and g_n be two sequences with exponential generating functions*

$$F(t) = \sum_{n=0}^{\infty}\frac{f_n t^n}{n!}, \quad G(t) = \sum_{n=0}^{\infty}\frac{g_n t^n}{n!}.$$

Show that the following are equivalent:

(a)

$$g_n = \sum_{k=0}^{n}\binom{n}{k} f_k;$$

(b)

$$G(t) = F(t)e^t.$$

Proof. Since an exponential generating function uniquely determines the sequence and conversely, the equivalence is immediate in view of the following calculation:

$$\begin{aligned} & g_n = \sum_{k=0}^{n} \binom{n}{k} f_k \\ \iff \quad & G(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} f_k \\ \iff \quad & G(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f_k \end{aligned}$$

Interchanging the order of summation:

$$\iff \quad G(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} f_k \sum_{n=k}^{\infty} \frac{t^{n-k}}{(n-k)!}$$

Making the change of variable $n - k = u$ in the inner sum:

$$\begin{aligned} \iff \quad & G(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} f_k \sum_{u=0}^{\infty} \frac{t^u}{u!} \\ \iff \quad & G(t) = F(t)e^t \end{aligned}$$

This finishes the proof. □

PROBLEM 3. *Let d_n be the number of derangements of an n-element set (that is, the number of elements of S_n without any fixed points). Show that*

$$d_n = nd_{n-1} + (-1)^n$$

Deduce that d_n is odd if and only if n is even.

Proof. We have shown in class that d_n satisfies the following recurrence:

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad \text{for } n > 1 \quad d_0 = 1, d_1 = 0.$$

Substracting nd_{n-1}, the relation becomes:

$$d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2})$$

Setting $c_n = d_n - nd_{n-1}$, we see that $c_n = -c_{n-1}$ with $c_1 = -1$. Thus, it follows immediately that $c_n = (-1)^n$ which proves the relation required.

Now, the final parity argument is easy: if n is even, then, $d_n \equiv (-1)^n = 1 \bmod 2$, so d_n is odd. On the other hand, if n is odd, then, d_n is even since $n - 1$ is even and we have just shown that d_{n-1} is odd; using the fact that $d_n = nd_{n-1} - 1$, we have that d_n is even. This finishes the problem. □

PROBLEM 4. *The **Bernoulli numbers** b_n are defined by the recurrence $b_0 = 1$ and*

$$\sum_{k=0}^{n} \binom{n+1}{k} b_k = 0$$

for $n \geqslant 1$. Prove that

$$f(t) := \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n = \frac{t}{e^t - 1}.$$

Proof. From the given recurrence, we have

$$b_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k$$

for $n > 0$. Now, we compute the function $f(t)$ as follows:

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{b_n t^n}{n!} \\ &= 1 - \sum_{n=1}^{\infty} \frac{t^n}{(n+1)!} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k \\ &= 1 - \sum_{n=1}^{\infty} t^n \sum_{k=0}^{n-1} \frac{1}{k!(n+1-k)!} b_k \end{aligned}$$

Put $m = n - 1$:

$$= 1 - \sum_{m=0}^{\infty} t^{m+1} \sum_{k=0}^{m} \frac{1}{k!(m+2-k)!} b_k$$

Interchanging the order of summation, we have:

$$= 1 - \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{t^{m+1}}{k!(m+2-k)!} b_k$$

Put $v = m - k + 2$ to get:

$$\begin{aligned} &= 1 - \sum_{k=0}^{\infty} \sum_{v=2}^{\infty} \frac{t^{v+k-1}}{k!v!} b_k \\ &= 1 - \sum_{k=0}^{\infty} \frac{t^{k-1} b_k}{k!} \sum_{v=2}^{\infty} \frac{t^v}{v!} \\ f(t) &= 1 - \frac{f(t)(e^t - t - 1)}{t} \\ f(t) &= \frac{t}{e^t - 1}. \end{aligned}$$

This finishes the proof. □

Problem 5. *In the previous question, show that*

$$f(t) + \frac{t}{2}$$

is an even function of t (that is, it is invariant under the map $t \mapsto -t$). Deduce that $b_n = 0$ for all odd $n \geqslant 3$.

Proof. To prove the first part, we simply compute:

$$\begin{aligned} f(-t) + \frac{-t}{2} &= \frac{-t}{e^{-t}-1} - \frac{t}{2} \\ &= \frac{2te^t - t(e^t-1)}{2(e^t-1)} \\ &= \frac{te^t + t}{2(e^t-1)} \\ &= \frac{t}{2}\left(\frac{e^t+1}{e^t-1}\right) \\ &= \frac{t}{2}\left(\frac{e^t-1+2}{e^t-1}\right) \\ &= \frac{t}{2}\left(1 + \frac{2}{e^t-1}\right) \\ &= \frac{t}{2} + \frac{t}{e^t-1} \\ &= f(t) + \frac{t}{2} \end{aligned}$$

Now, plugging our power series for $f(t)$, we get:

$$\sum_{n=0}^{\infty} \frac{(-1)^n b_n t^n}{n!} + \frac{-t}{2} = \sum_{n=0}^{\infty} \frac{b_n t^n}{n!} + \frac{t}{2}$$

Comparing the coefficient of t^n on both sides implies that $b_1 = -1/2$ and that $b_n = 0$ for all odd $n \geqslant 3$. □

Problem 6. *Let $s(n,k)$ and $S(n,k)$ denote the Stirling numbers of the first and second kind respectively. Show that $S(n,k) \leqslant |s(n,k)|$. Deduce that the n-th Bell number satisfies $B_n \leqslant n!$.*

Proof. By definition, we have the following descriptions:

$S(n,k)$**:** the number of (unordered) partitions of $[n]$ into k blocks.
$|s(n,k)|$**:** the number of elements in S_n that can be written as a product of k disjoint cycles.
B_n**:** the number of (unordered) partitions of $[n]$.

Based on these descriptions, it is easy to deduce the claim about Bell numbers: indeed, we have

$$\begin{aligned} B_n &= \sum_{k=1}^{n} S(n,k) \\ &\leqslant \sum_{k=1}^{n} |s(n,k)| = n!. \end{aligned}$$

Thus, to finish the problem, we must show that $S(n,k) \leqslant |s(n,k)|$. We show that there is an injection from the set $\Sigma(n,k)$ of partitions of $[n]$ into k blocks into the set of elements in S_n that can be written as a product of k disjoint cycles. Indeed, given an element $\sigma = \sigma[1] \coprod \cdots \coprod \sigma[k] \in \Sigma(n,k)$, we may construct a permutation

$\dot{\sigma}$ as follows:

$$\dot{\sigma} = (\sigma_{11} \dots \sigma_{1r_1}) \dots (\sigma_{k1} \dots \sigma_{kr_k})$$

where $\sigma_{j1} < \dots < \sigma_{jr_j}$ and $\sigma[j] = \{\sigma_{j1}, \dots, \sigma_{jr_j}\}$. This map is clearly an injection and this shows that $S(n,k) \leqslant |s(n,k)|$. □

Problem 7. *With B_n denoting the n-th Bell number (that is, the number of partitions of an n-element set), show that*

$$B_n = \sum_{k=1}^{n} \frac{k^n}{k!} \left(\sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \right).$$

Proof. We begin by noting

$$B_n = \sum_{k=1}^{n} S(n,k)$$

where $S(n,k)$ is the Stirling number of the second kind. Thus (for $n > 0$):

$$= \sum_{k=1}^{n} \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n.$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \frac{1}{i!(k-i)!} (k-i)^n.$$

Substituting $j = k - i$, it follows that:

$$= \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k-j} \frac{1}{(k-j)!j!} j^n$$

Interchanging the order of summation, it follows that:

$$= \sum_{j=1}^{n} \sum_{k=j}^{n} (-1)^{k-j} \frac{1}{(k-j)!j!} j^n$$

$$= \sum_{j=1}^{n} \frac{j^n}{j!} \sum_{k=j}^{n} (-1)^{k-j} \frac{1}{(k-j)!}$$

Making a change of variable in the above sum, it follows that:

$$= \sum_{j=1}^{n} \frac{j^n}{j!} \sum_{k=1}^{n-j} (-1)^k \frac{1}{k!}$$

This finishes the proof. □

Problem 8. *A permutation σ of the symmetric group S_n is called an* involution *if $\sigma^2 = 1$. Let $s(n)$ denote the number of involutions of S_n. Defining $s(0) = 1$, prove that*

$$\sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} = \exp\left(t + \frac{t^2}{2}\right).$$

Proof. We first establish a recurrence relation for $s(n)$. Let $\sigma \in S_n$ such that $\sigma^2 = 1$. If $\sigma(n) = n$, then, σ is an involution in S_{n-1}. Otherwise, $(n\sigma(n))\sigma$ is an involution on $n-2$ symbols. This gives us the following recurrence:

$$s(n) = s(n-1) + (n-1)s(n-2).$$

Let $S(t) = \sum_{n=0}^{\infty} \frac{s(n)t^n}{n!}$. We now do the following:

$$\begin{aligned}
(1+t)S(t) &= (1+t)\sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} + \sum_{n=0}^{\infty} \frac{s(n)t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} + \sum_{n=1}^{\infty} \frac{s(n-1)t^n}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{s(n)t^n}{n!} + \sum_{n=1}^{\infty} \frac{ns(n-1)t^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{s(n)+ns(n-1)}{n!} t^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{s(n+1)}{n!} t^n \\
&= 1 + \sum_{n=2}^{\infty} \frac{ns(n)t^{n-1}}{n!} \\
&= \sum_{n=1}^{\infty} \frac{ns(n)t^{n-1}}{n!} \\
&= S'(t)
\end{aligned}$$

This proves that $S(t) = \exp(t + t^2/2)$ since $S(0) = 1$. □

Problem 9. *Prove that*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+n-i}{k-i} = \binom{m}{k}$$

if $k \leqslant m$ and 0 otherwise. [Hint: if you have m red cards and n blue cards, count the number of k-element subsets consisting solely of red cards.]

Proof. Continuing the notation in Hint, let A denote the set of all k-subsets of $m+n$ cards. Index the blue cards from 1 to n. For each $i \in [n]$, let A_i denote the set of k-subsets of $m+n$ cards which consists of the ith blue card.

The number of k-subsets of $m+n$ cards which consists only of red cards is then given by:

$$\left| A \setminus \bigcup_{i=1}^{n} A_i \right| = \begin{cases} \binom{m}{k}, & \text{if } k \leqslant m \\ 0, & \text{otherwise} \end{cases}.$$

Applying the Principle of Inclusion-Exclusion, we see:

$$\left| A \setminus \bigcup_{i=1}^{m} A_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} A_I \tag{1}$$

where $A_I = \cap_{i \in I} A_i$ (with the obvious convention that $A_\varnothing = A$). Clearly, A_I is just the set of k-subsets of $m+n$ cards which contains all the blue cards whose indices are in I. If $|I| = i$, then, this number is just $\binom{m+n-i}{k-i}$. Notice that this number is independent of the subset I (depends only on $|I|$). Now, continuing from (1):

$$\begin{aligned} \left| A \setminus \bigcup_{i=1}^{m} A_i \right| &= \sum_{I \subseteq [n]} (-1)^{|I|} A_I \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+n-i}{k-i}. \end{aligned}$$

This completes the proof. □

Problem 10. *Let A_n denote the number of maps $f : [n] \to [n]$ with the property that if i lies in the image of f, then, so does $[i]$. Prove that*

$$A_n = \sum_{k=1}^{n} \binom{n}{k} A_{n-k}$$

Deduce that

$$A_n = \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}}$$

In this problem, we have (by convention, for the scholars of the empty set) that $A_0 = 1$.

Proof. We reinterpret the number A_n. Firstly, a function $f : [n] \to [n]$ that satisfies the hypothesis of the problem must surject onto $[k]$ for some $1 \leqslant k \leqslant n$ and conversely; now, a surjective function $f : [n] \to [k]$ determines an ordered partition of $[n]$ into k blocks and conversely, given a partition

$$\mathcal{P} : \qquad S_1 \coprod S_2 \coprod \cdots \coprod S_k = [n]$$

of $[n]$ into k blocks, we may construct $f_{\mathcal{P}} : [n] \to [k]$ by $f_{\mathcal{P}}(j) = \ell$ if $j \in S_\ell$.

Thus, A_n is the number of ordered partition of $[n]$ into non-empty blocks (for $n > 0$); let $\mathcal{A}_n$ denote the set of ordered partitions of $[n]$. Given an ordered partition $\mathcal{P}$ of $[n]$, we write $\mathcal{P}(1)$ for the first block of the given partition. Then, we have the following count (assuming that $n > 0$):

$$\begin{aligned} A_n &= \sum_{\mathcal{P} \in \mathcal{A}_n} 1 \\ &= \sum_{\varnothing \neq S \subseteq [n]} \sum_{\mathcal{P}[1] = S} 1 \end{aligned}$$

The inner sum counts the number of ordered partitions with first block S which is the same as the number of ordered partitions of $[n - |S|]$:

$$= \sum_{\varnothing \neq S \subseteq [n]} A_{n-|S|}$$
$$= \sum_{k=1}^{n} \binom{n}{k} A_{n-k}$$

This finishes the proof of the first equality. Now to prove the second equality, we proceed by induction on n. For $n = 0$, the result holds. Indeed, the sum defining A_0 is a geometric series with initial term $1/2$ and common ratio $1/2$ whence its sum is 1 (by convention, as always, $0^0 = 1$). Since the recursion holds for all $n \geqslant 1$, we plug this explicit formula into the recursion and get that:

$$A_n = \sum_{k=1}^{n} \binom{n}{k} A_{n-k}$$
$$= \sum_{k=1}^{n} \binom{n}{k} \sum_{\ell=0}^{\infty} \frac{\ell^{n-k}}{2^{\ell+1}}$$

Interchanging the order of summation:

$$= \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell+1}} \sum_{k=1}^{n} \binom{n}{k} \ell^{n-k}$$

We keep the term $\ell = 0$ out (and implement our convention that $0^0 = 1$):

$$= \frac{1}{2} + \sum_{\ell=1}^{\infty} \frac{\ell^n}{2^{\ell+1}} \sum_{k=1}^{n} \binom{n}{k} \ell^{-k}$$
$$= \frac{1}{2} + \sum_{\ell=1}^{\infty} \frac{\ell^n}{2^{\ell+1}} \left(\left(1 + \frac{1}{\ell}\right)^n - 1 \right)$$
$$= \frac{1}{2} + \sum_{\ell=1}^{\infty} \frac{(\ell+1)^n}{2^{\ell+1}} - \sum_{\ell=1}^{\infty} \frac{\ell^n}{2^{\ell+1}}$$

Making a change of variable in the first sum:

$$= \frac{1}{2} + \sum_{\ell=2}^{\infty} \frac{\ell^n}{2^{\ell}} - \sum_{\ell=1}^{\infty} \frac{\ell^n}{2^{\ell+1}}$$
$$= \sum_{\ell=1}^{\infty} \frac{\ell^n}{2^{\ell+1}}$$

This completes the problem. $\square$