Math 802 - Combinatorics	Fall 2015 - November 26, 2015
Assignment 3	*François Séguin*

Question 1

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two locally finite posets. Show that

$$\mu((x_1, y_1), (x_2, y_2)) = \mu(x_1, x_2)\mu(y_1, y_2).$$

Solution

First, note that since both P_1 and P_2 are locally finite posets, then so is $P_1 \times P_2$ as the product poset. Indeed,

$$[(x_1, y_1), (x_2, y_2)] = \{(x, y) | x_1 \leq x \leq x_2 \text{ and } y_1 \leq y \leq y_2\} = [x_1, x_2] \times [y_1, y_2]$$

as sets, and so is finite.

Now we will use an induction argument to show the desired statement. Fix (x_1, y_1) in the poset $P_1 \times P_2$. We will induce on (x_2, y_2). Note that whenever $(x_1, y_1) \not\leq (x_2, y_2)$, both sides of the desired equation are trivially equal to 0 (as one of $\mu(x_1, x_2)$ and $\mu(y_1, y_2)$ is zero).

For the base case, consider $(x_2, y_2) = (x_1, y_1)$. Then, by definition,

$$\mu((x_1, y_1), (x_1, y_1)) = 1 = \mu(x_1, x_1)\mu(y_1, y_1).$$

Now suppose that the statement is true for $(x, y) < (x_2, y_2)$ (that is $(x, y) \leq (x_2, y_2)$ and $(x, y) \neq (x_2, y_2)$).

First recall that by definition, in any poset, if $\alpha \neq \gamma$ then

$$\sum_{\alpha \leq \beta \leq \gamma} \mu(\alpha, \gamma) = 0$$

that is

$$\sum_{\alpha \leq \beta < \gamma} \mu(\alpha, \gamma) = -\mu(\alpha, \beta).$$

Getting back to the case at hand, since $(x_1, y_1) \neq (x_2, y_2)$ here,

$$\begin{aligned} 0 &= \sum_{(x_1, y_1) \leq (x, y) \leq (x_2, y_2)} \mu((x_1, y_1), (x, y)) \\ &= \sum_{\substack{x_1 \leq x \leq x_2 \\ y_1 \leq y \leq y_2}} \mu((x_1, y_1), (x, y)). \end{aligned}$$

Now we split the sum to get

$$0 = \underbrace{\sum_{\substack{x_1 \le x < x_2 \\ y_1 \le y < y_2}} \mu((x_1, y_1), (x, y))}_{S_1} + \underbrace{\sum_{x_1 \le x < x_2} \mu((x_1, y_1), (x, y_2))}_{S_2} \tag{1}$$
$$+ \underbrace{\sum_{y_1 \le y < y_2} \mu((x_1, y_1), (x_2, y))}_{S_3} + \mu((x_1, y_1), (x_2, y_2)).$$

Since all the summands in the three sums above satisfy the Induction hypothesis, we have that

$$\begin{aligned} S_1 &= \sum_{\substack{x_1 \le x < x_2 \\ y_1 \le y < y_2}} \mu(x_1, x)\mu(y_1, y) \\ &= \left(\sum_{x_1 \le x < x_2} \mu(x_1, x) \right) \left(\sum_{y_1 \le y < y_2} \mu(y_1, y) \right) \end{aligned}$$

and by the remark above

$$\begin{aligned} &= (-\mu(x_1, x_2))\,(-\mu(y_1, y_2)) \\ &= \mu(x_1, x_2)\mu(y_1, y_2). \end{aligned}$$

Also,

$$\begin{aligned} S_2 &= \sum_{x_1 \le x < x_2} \mu(x_1, x)\mu(y_1, y_2) \\ &= \mu(y_1, y_2) \sum_{x_1 \le x < x_2} \mu(x_1, x) \\ &= -\mu(x_1, x_2)\mu(y_1, y_2) \end{aligned}$$

and similarly,

$$\begin{aligned} S_3 &= \sum_{y_1 \le y < y_2} \mu(x_1, x_2)\mu(y_1, y) \\ &= -\mu(x_1, x_2)\mu(y_1, y_2) \end{aligned}$$

plugging everything back in (1), we get,

$$\begin{aligned} 0 = {} & \mu(x_1, x_2)\mu(y_1, y_2) - \mu(x_1, x_2)\mu(y_1, y_2) \\ & - \mu(x_1, x_2)\mu(y_1, y_2) + \mu((x_1, y_1), (x_2, y_2)) \end{aligned}$$

and so we get

$$\mu(x_1, x_2)\mu(y_1, y_2) = \mu((x_1, y_1), (x_2, y_2)).$$

Question 2

Find a formula for the number of sequences of length n using the symbols A, B, C, D which have the symbols A, B, C appearing at least once.

Solution

This is a simple application of the Inclusion-Exclusion principle.

Let S be the set of all sequences of length n using the symbols A, B, C, D. Let S_A be those sequences in S which do not use the symbol A, and similarly for S_B and S_C. Finally, for $L \subseteq \{A, B, C\}$, define $S_L = \bigcap_{\star \in L} S_\star$ (and as usual $S_\varnothing = S$). Then, the desired quantity is simply expressed by

$$N(n) := \left| S \setminus \bigcup_{\star \in \{A,B,C\}} S_\star \right| = \sum_{L \subseteq \{A,B,C\}} (-1)^{|L|} |S_L|$$

First note that $|S| = 4^n$. Now when L consist of one symbol, $|S_L|$ is the number of sequences where this symbol does not appear, which is clearly 3^n (the number of sequences on three symbols). Similarly, if L contains two symbols, $|S_L| = 2^n$ as it is the number of sequences on two symbols. Finally, if $L = \{A, B, C\}$, then $|S_L| = 1 = 1^n$ as the only sequence of the character D is $(D, D, \ldots, D)$.

We thus have

$$\begin{aligned} \sum_{L \subseteq \{A,B,C\}} (-1)^{|L|} |S_L| &= \sum_{i=0}^{3} \binom{3}{i} (-1)^i (4-i)^n \\ &= 4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1. \end{aligned}$$

Therefore, the desired formula is

$$4^n - 3^{n+1} + 3 \cdot 2^n - 1.$$

Question 3

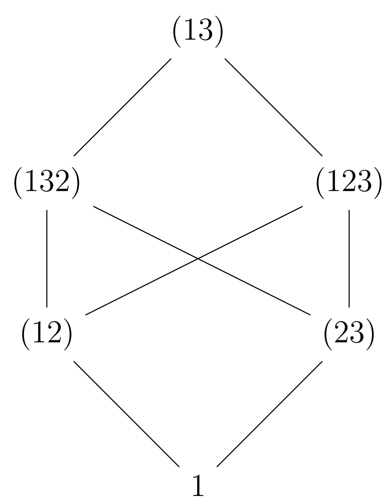
Draw the Hasse diagram for S_3 with the Bruhat order and determine completely the Möbius function on this poset.

Solution

We define the Bruhat order on S_n as follows. A permutation σ covers a permutation τ if and only if $\sigma(k) = \tau(k)$ for all $k \in [n]$ except two indices i and j with $i < j$ where $\sigma(i) > \tau(i)$ (and there is no proper chain of covering between σ and τ). In other words, the table format of permutations σ and τ are identical except for two entries on the bottom lines which are swapped. Look at the number appearing in the first of these two entries; the permutation having the largest number there covers the other one. From this definition, we list the elements of S_3 in the table below and look for the possible coverings.

σ	Table format	σ covers ...	with inversion $(i, j) = \ldots$
1	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\varnothing$	
(12)	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	1	$(1, 2)$
(23)	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	1	$(2, 3)$
(123)	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	(12)	$(2, 3)$
		(23)	$(1, 3)$
(132)	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	(12)	$(1, 3)$
		(23)	$(1, 2)$
(13)	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	(123)	$(1, 2)$
		(132)	$(2, 3)$

From this, we see that the corresponding Hasse diagram is the following.



To compute the Möbius function for this poset, we start by considering $\mu(1, \cdot)$. Looking at the Hasse diagram above, it is easy to see that the Möbius function in this case is given in the following table.

<table>
<tr><th>$\cdot$</th><th>$\mu(1,\cdot)$</th></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>(12)</td><td rowspan="2">-1</td></tr>
<tr><td>(23)</td></tr>
<tr><td>(123)</td><td rowspan="2">1</td></tr>
<tr><td>(132)</td></tr>
<tr><td>(13)</td><td>-1</td></tr>
</table>

Furthermore, we can also see easily that for both $\sigma = (12)$ and (23), we have that $\mu(\sigma,(123)) = \mu(\sigma,(132)) = -1$ and $\mu(\sigma,(13)) = 1$. Finally, we have $\mu((123),(13)) = \mu((132),(13)) = -1$.

The above combined with the fact that $\mu(\sigma,\sigma) = 1$ for any $\sigma \in S_3$ determines the Möbius function on any element of this poset.

Question 4

Let G be a group acting on a set X and H a group acting on a set Y. Assume that X and Y are disjoint and let $U = X \coprod Y$. For $g \in G, h \in H$ and $z \in U$, define

$$(g,h)\cdot z = \begin{cases} g\cdot z & \text{if } z \in X \\ h \cdot z & \text{if } z \in Y. \end{cases}$$

Show that this defines an action of $G \times H$ on U.

Solution

We simply need to verify that the two axioms of a group action.

In $G \times H$, the identity is simply $(1_G, 1_H)$. Therefore we have that for any $z \in U$,

$$\begin{aligned} (1_G,1_H)\cdot z &= \begin{cases} 1_G\cdot z & \text{if } z \in X \\ 1_H \cdot z & \text{if } z \in Y \end{cases} \\ &= \begin{cases} z & \text{if } z \in X \\ z & \text{if } z \in Y \end{cases} \\ &= z. \end{aligned}$$

For the next axiom, first note that for any $(g,h) \in G \times H$, $(g,h)\cdot z \in X \Leftrightarrow z \in X$ and similarly $(g,h)\cdot z \in Y \Leftrightarrow z \in Y$. Now let $(g_1,h_1),(g_2,h_2) \in G \times H$, and $z \in U$. We have

$$(g_1,h_1)\cdot[(g_2,h_2)\cdot z)] = \begin{cases} (g_1,h_1)\cdot(g_2\cdot z) & \text{if } z \in X \\ (g_1,h_1)\cdot(h_2\cdot z) & \text{if } z \in Y \end{cases}$$

By the above remark, the first case also implies that $(g_2 \cdot z) \in X$ and so g_1 is applied to $g_2 \cdot z$, and similarly for the second case. We have

$$= \begin{cases} g_1 \cdot (g_2 \cdot z) & \text{if } z \in X \\ h_1 \cdot (h_2 \cdot z) & \text{if } z \in Y \end{cases}$$

Since both the actions of G on X and H on Y satisfy the axioms of group action, we have

$$\begin{aligned} &= \begin{cases} (g_1 g_2) \cdot z & \text{if } z \in X \\ (h_1 h_2) \cdot z & \text{if } z \in Y \end{cases} \\ &= (g_1 g_2, h_1 h_2) \cdot z \\ &= [(g_1, h_1) \cdot_{G \times H} (g_2, h_2)] \cdot z \end{aligned}$$

and thus both axioms of group action are satisfied.

Question 5

Let G be a finite group acting on a finite set X. Show that if G acts transitively on X, then

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| = 1.$$

Solution

Recall the following lemma.

LEMMA 1 (Burnside's Lemma). *Let G be a finite group acting on a finite set X. Then,*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

where $|X/G|$ denote the number of orbits of X under the action of G.

From this, we see that we only need to prove that the number of orbit of X when G acts transitively is 1.

Fix an element $x \in X$. G acting transitively means that for any element $y \in X$, there exists a $g \in G$ such that $y = g \cdot x$. In other words, every element of X is in the orbit of x. That is to say that $Gx = X$, and since any action partitions X in non-empty disjoint orbits, it is clear that there can only be 1 orbit in this case. Burnside's Lemma then implies that

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| = 1$$

as required.

Question 6

Show that the number of permutations of type $(c_1, c_2, \ldots, c_n)$ in S_n, denoted $p_n(c_1, \ldots, c_n)$ is

$$\frac{n!}{1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!}$$

when $\sum_{i=1}^n i \cdot c_i = n$, and 0 otherwise.

Solution

We start by fixing $(c_1, c_2, \ldots, c_n)$ such that $\sum_{i=1}^n i \cdot c_i = n$. If this last condition is not satisfied, then we would need more than n elements to permute and there is no permutation of such cycle type in S_n.

Then, notice that given any permutation of this cycle type, we can order its cycles by length and write it as

$$\underbrace{(\alpha_1)\cdots(\alpha_{c_1})}_{1-\text{cycles}}\underbrace{(\alpha_{c_1+1}\alpha_{c_1+2})\cdots(\alpha_{c_1+c_2-1}\alpha_{c_1+c_2})}_{2-\text{cycles}}\cdots\underbrace{\cdots(\alpha_{n-k}\cdots\alpha_{n-1}\alpha_n)}_{k-\text{cycles}}$$

Since the cycle type is fixed, the parentheses add no information to the above and given an ordering of $[n]$, there is such a permutation of this cycle type. Note that there are exactly $n!$ such ordering of $[n]$.

However, we have counted each permutation more than once as two such ordering might give rise to the same permutation. Namely, for the l-cycles, there are $c_l!$ ways to permute the cycles between themselves without changing the permutation. Also, for each cycle, there are l ways to write it by cycling it (e.g. $(123) = (231) = (312)$). Putting all of this together, there are $l^{c_l}c_l!$ ways to modify the ordering of $[n]$ with respect to the l-cycles, while giving rise to the same permutation. Since this holds for every cycle length, we conclude that we counted each permutation exactly $1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!$ times. Thus, $p_n(c_1, \ldots, c_n)$ is

$$\frac{n!}{1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!}.$$

Question 7

Calculate the cycle index polynomial $P_{S_3}(x_1, x_2, x_3)$ where S_3 acts on $[3]$ in the usual way.

Solution

We begin by listing all elements of S_3 along with their cycle type.

Note that since the action of S_3 is the usual action, its association with S_3 is the identity and the cycle type is simply the cycle type of the original element in S_3.

σ	(c_1, c_2, c_3)
1	$(3, 0, 0)$
(12)	$(1, 1, 0)$
(13)	$(1, 1, 0)$
(23)	$(1, 1, 0)$
(123)	$(0, 0, 1)$
(132)	$(0, 0, 1)$

Thus, the cycle index polynomial is simply

$$\begin{aligned} P_{S_3}(\underline{x}) &= \frac{1}{6} \sum_{\sigma \in S_3} \underline{x}^{\sigma} \\ &= \frac{1}{6} \left(x_1^3 + 3x_1x_2 + 2x_3 \right). \end{aligned}$$

Question 8

Let G and H be finite groups acting on finite sets X and Y respectively. In Question 4, we defined an action of $G \times H$ on $X \coprod Y$. If P_G and P_H indicate the cycle index polynomial of G acting on X and H acting on Y respectively, show that the cycle index polynomial $P_{G\times H}$ of $G \times H$ acting on $X \coprod Y$ is $P_G P_H$.

Solution

First call $n = |X|$, $m = |Y|$ and without loss of generality, let $n \geq m$. Let us look at $P_{G\times H}$. We have by definition that

$$\begin{aligned} P_{G\times H}(\underline{x}) &= \frac{1}{|G|\,|H|} \sum_{(g,h) \in G\times H} \underline{x}^{(g,h)} \\ &= \frac{1}{|G|\,|H|} \sum_{g \in G} \sum_{h \in H} \underline{x}^{(g,h)}. \end{aligned}$$

Now consider the monomial $\underline{x}^{(g,h)}$. Of all the elements of $X \coprod Y$, (g, h) acts on those of X the same way g would and on those of Y the same way h would. Thus, the cycle decomposition of (g, h) as an element of $\mathrm{Sym}(X \coprod Y)$ is simply a concatenation of the cycles of g in $\mathrm{Sym}(X)$ with the cycles of h in $\mathrm{Sym}(Y)$. Hence, the number of k-cycles in (g, h) is the number of k-cycles in g, say $c_k(g)$, plus the number of k-cycles in h, say $c_k(h)$. Therefore,

$$\underline{x}^{(g,h)} = x_1^{c_1(g)+c_1(h)} x_2^{c_2(g)+c_2(h)} \cdots x_n^{c_n(g)+c_n(h)}$$

where obviously $c_{m+1}(h) = \cdots = c_n(h) = 0$. So

$$= \underline{x}^{g} \underline{x}^{h}.$$

Note in particular that even though a priori the monomial $\underline{x}^{(g,h)}$ is composed of nm variables, only the first n variables $x_1, \ldots, x_n$ will appear.

We then have that

$$
\begin{aligned}
P_{G\times H}(\underline{x}) &= \frac{1}{|G|\,|H|}\sum_{g\in G}\sum_{h\in H}\underline{x}^{(g,h)} \\
&= \frac{1}{|G|\,|H|}\sum_{g\in G}\sum_{h\in H}\underline{x}^{g}\underline{x}^{h} \\
&= \left(\frac{1}{|G|}\sum_{g\in G}\underline{x}^{g}\right)\left(\frac{1}{|H|}\sum_{h\in H}\underline{x}^{h}\right) \\
&= P_G(\underline{x})P_H(\underline{x}).
\end{aligned}
$$

Question 9

Let p be a prime number. Show that the number of $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ with determinant not divisible by p is given by

$$(p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}).$$

Solution

First note that as p is prime, $\mathbb{Z}/p\mathbb{Z}$ is a field with p elements $\mathbb{F}_p$. Therefore, the set of $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ with determinant not divisible by p is the set of $n\times n$ matrices in $\mathbb{F}_p$ with non-zero determinant in the field, i.e. it is simply the ring $GL_n(\mathbb{F}_p)$. Given a $n\times n$ matrix over a field K, we know that its determinant is non-zero if and only if its columns are linearly independent, non-zero vectors in the vector field K^n.

Here, the question amounts to counting the ways to write n vectors of $\mathbb{F}_p^n$ that are linearly independent.

For the first vector, there are $p^n - 1$ choices as any vector except for the zero vector works.

For the second choice, we have $p^n - p$ possibilities as anything but $\mathbb{F}_p$ multiples of the first vector works, and there are p of the latter.

For the third vector, we can chose anything that is not a $\mathbb{F}_p$ linear combination of the previous two vectors. Since there are $p \times p = p^2$ such linear combinations, the number of ways to choose the third vector is $p^n - p^2$.

We can continue this process for each of the n vectors, noting that for the $(k+1)^{\text{th}}$ vector, there are p^k $\mathbb{F}_p$-linear combinations of the previous k vectors, and so there are $p^n - p^k$ ways to choose this one. We conclude that the total number of $n \times n$ matrices with non-zero determinant in $\mathbb{F}_p$ is

$$(p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}).$$

Question 10

Let S_n act on $[n]$ in the usual way. Let P_{S_n} be the cycle index polynomial. Prove that P_{S_n} is the coefficient of z^n in the power series expansion of

$$\exp(zx_1 + z^2x_2/2 + z^3x_3/3 + \cdots).$$

Solution

First note that

$$\begin{aligned} P_{S_n}(\underline{x}) &= \frac{1}{n!}\sum_{\sigma\in S_n} \underline{x}^\sigma \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} x_1^{c_1(\sigma)}x_2^{c_2(\sigma)}\cdots x_n^{c_n(\sigma)}. \end{aligned}$$

Since any $\sigma \in S_n$ has some cycle type, we can instead sum over all the possible cycle types. Using the notation from Question 6,

$$\begin{aligned} &= \frac{1}{n!}\sum_{(c_1,\dots,c_n)} p_n(c_1,\dots,c_n)x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n} \\ &= \frac{1}{n!}\sum_{\substack{(c_1,\dots,c_n)\\ \sum_i i\cdot c_i = n}} \frac{n!}{1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!}x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n} \\ &= \sum_{\substack{(c_1,\dots,c_n)\\ \sum_i i\cdot c_i = n}} \frac{x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}}{1^{c_1}c_1!2^{c_2}c_2!\cdots n^{c_n}c_n!}. \end{aligned}$$

Now consider

$$\exp(zx_1 + z^2x_2/2 + z^3x_3/3 + \cdots) = \prod_{n=1}^{\infty}\exp\left(\frac{z^n x_n}{n}\right).$$

Expanding each factor in its power series, we get

$$\begin{aligned} &= \prod_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\left(\frac{z^n x_n}{n}\right)^k \frac{1}{k!}\right) \\ &= \prod_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{z^{nk}x_n^k}{n^k k!}\right). \end{aligned}$$

When trying to expand this product, we will need to pick exactly one term of the sum for each factor of the product. Suppose that in the i^{th} factor we pick the c_i^{th} term of the sum.

Then, the resulting term will have a z^n precisely if $\sum_{i=1}^{\infty} i \cdot c_i = n$. Now note that this forces all c_i with $i > n$ to be zero. From this, we see that the coefficient of z^n will be the sum over all possible choices of $(c_1, \ldots, c_n)$ satisfying the condition stated, that is

$$\sum_{\substack{(c_1,\ldots,c_n) \\ \sum_i i \cdot c_i = n}} \frac{x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!}$$

which is exactly P_{S_n} as calculated above.