Math 801 - Graph Theory	Fall 2014 - November 19, 2014
Assignment 4	*François Séguin*

Question 1

PROPOSITION 1. *A graph X is 2-colorable if and only if it is bipartite.*

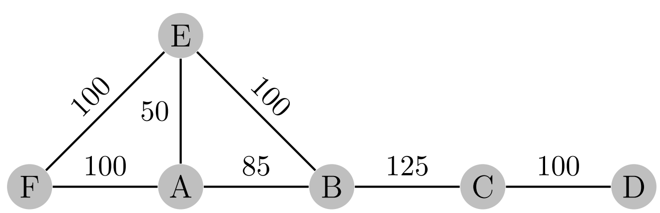
Proof. If a graph X is 2-colorable, we can color it using colors a and b. Grouping vertices of color a in A and those of color b in B, we obtain two sets inside which no two vertices are adjacent. Conversely, if X is Bipartite, we can group the vertices in sets A and B. Simply color all vertices of A with color a and vertices of B with color b to obtain a proper 2-coloring. $\square$

Remark 1. If X is a graph with chromatic number 2, then it is 2-colorable, and therefore it is bipartite.

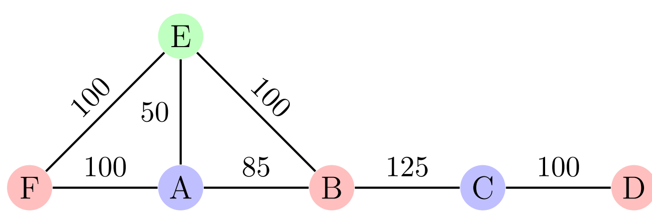
Question 2

Let the graph X have vertex set $\{A, B, C, D, E, F\}$ and connect two vertices by an edge if the two corresponding television stations are within 150 miles of each other. Then, the minimal number of frequencies needed correspond to the chromatic number of X.

The graph X looks like the following.



Notice first that the stations A, B and E form a 3-clique, i.e. induce K_3 as a subgraph. Therefore, we know we need at least 3 colors to color this graph. However, consider the following 3-coloring.



That is we have nodes A and C blue, F, B and D red and E green. This is a proper 3-coloring of the graph X. We conclude that $\chi(X) = 3$ and we thus need 3 frequencies.

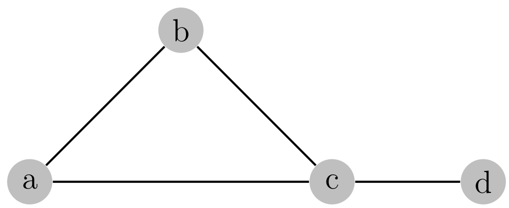
Question 3

Proposition 2. *Let X be a graph with at least one edge. The sum of the coefficients of the polynomial $P_X(\lambda)$ is zero.*

Proof. Remark that the polynomial $P_X(\lambda)$ counts the number of ways to color the graph X using λ colors. The sum of the coefficients of $P_X(\lambda)$ is given by $P_X(1)$, which in turn corresponds to the number of ways to color the graph X with 1 color. However, since X has at least one edge, the endpoints of this edge cannot be colored using the same color, and thus the number $P_X(1)$ is zero. □

Question 4

We want to compute the chromatic polynomial for the following graph X.



First notice that the graph K_3 is realized here as the subgraph induced by vertices $\{a, b, c\}$. The chromatic polynomial for this graph is clearly $P_{K_3}(\lambda) = \lambda(\lambda - 1)(\lambda - 2)$ as there are λ ways of coloring a first vertex, $\lambda - 1$ ways of coloring an adjacent vertex and then only $\lambda - 2$ colors are left for the third vertex.

Also, given a coloring of $\{a, b, c\}$, there are $\lambda - 1$ ways to extend this coloring to X as we simply color vertex d any color other than the color of c. Therefore, the chromatic polynomial for X is given by $P_X(\lambda) = \lambda(\lambda - 1)^2(\lambda - 2)$.

Question 5

Proposition 3. *Given two graphs X and Y, $\chi(X \vee Y) = \chi(X) + \chi(Y)$, where $X \vee Y$ denotes the join of the two graphs.*

Proof. First, given a $\chi(X)$ coloring on X and a $\chi(Y)$ coloring on Y, we can take the disjoint union of the two coloring (i.e. giving new names to the colors so that no color is used for both colorings) to find a $\chi(X) + \chi(Y)$ coloring of $X \vee Y$. This is a proper coloring as it is proper in X and in Y, and any color used in one of the graph is not used in the other. We then have that $\chi(X \vee Y) \leq \chi(X) + \chi(Y)$.

Now suppose that we have a proper coloring of $X \vee Y$ using t colors for $t < \chi(X) + \chi(Y)$. Given any two vertices $x \in X$ and $y \in Y$, x and y cannot share the same color as they are adjacent in $X \vee Y$. Therefore, the colors used for the vertices of X cannot appear as a color of the vertices of Y and vice versa. Hence, the number of colors used for the vertices of X plus the number of colors used for the vertices of Y is t. But as this coloring for $X \vee Y$ can be restricted to each of X and Y to yield a proper coloring of X and Y respectively, we know that $\chi(X) + \chi(Y) \leq$ #colors used for X + #colors used for $Y = t$ which is a contradiction of the hypothesis.

We therefore conclude that $\chi(X \vee Y) = \chi(X) + \chi(Y)$. □

Question 6

We want to compute the chromatic polynomial of the wheel graph $K_1 \vee C_n$. We do so by calculating, given λ colors, how many ways we can color this graph.

First, it is clear that there are λ ways to color the inner vertex (corresponding to K_1). Also, since this vertex is adjacent to every vertex in the subgraph C_n, this color cannot by used anymore. Therefore, we have $\lambda - 1$ colors to color C_n. The number of ways of doing so is simply given by $P_{C_n}(\lambda - 1)$. Since we already know the chromatic polynomial of C_n to be $(\lambda - 1)^n + (-1)^n(\lambda - 1)$, we have that

$$P_{K_1 \vee C_n} = \lambda \, P_{C_n}(\lambda - 1) = \lambda \left[(\lambda - 2)^n + (-1)^n (\lambda - 2) \right].$$

Question 7

THEOREM 4. *Let X be a connected graph with n vertices and $P_X(\lambda)$ be its chromatic polynomial. Then, for every $k \in \mathbb{N}^+$,*

$$P_X(k) \leq k(k-1)^{n-1}.$$

Proof. Since X is connected, there is a spanning tree T in X. Since T spans X, it is a tree on n vertices and therefore its chromatic polynomial is $P_T(\lambda) = \lambda(\lambda - 1)^{n-1}$.

Given any $k \in \mathbb{N}^+$, $P_X(k)$ gives the number of ways to color the graph X using k colors. However, any coloring of X is also a coloring of the spanning tree T as T is simply the graph X to which some edges were removed. Therefore, there are at least as many colorings of T as there are colorings X using k colors. Since the number of ways to color T using k colors is given by $P_T(k)$, we can rewrite the above as

$$P_X(k) \leq P_T(k) = k(k-1)^{n-1}.$$

□

Question 8

To begin, we recall the following theorem seen in class.

Theorem 5. *Let X be a loopless graph and e an edge of X. Then,*

$$P_X(\lambda) = P_{X-e}(\lambda) - P_{X/e}(\lambda)$$

where P_Y is the chromatic polynomial of graph Y.

Also, recall the following corollary of Theorem 5.

Theorem 6. *Let X be a loopless graph with n vertices. Then,*

$$P_X(\lambda) = \lambda^n - |E(X)| \lambda^{n-1} + \cdots$$

with the coefficients alternating in sign.

We now prove the following using a similar induction argument.

Theorem 7. *Let X be a connected simple graph with n vertices. Then $P_X(\lambda) = \lambda f(\lambda)$ where $f(0) \neq 0$. Moreover, the sign of $f(0)$ is given by $\operatorname{sgn}(f(0)) = (-1)^{n-1}$.*

Proof. We show this using induction on the number of edges of X.

If X has no edges, then as it is connected we need to have $n = 1$ and so X consists of a single vertex. Its chromatic polynomial is simply λ and so we have the result for $f(\lambda) = 1$. Also, since $n = 1$, $f(0) = 1$ is positive as wanted.

Suppose the result for graphs with less than K edges, and let X have $K \geq 1$ edges. We consider two cases.

Case 1. Suppose that X is a tree. Then, the chromatic polynomial of X is $P_X(\lambda) = \lambda(\lambda - 1)^{n-1}$. Taking $f(\lambda) = (\lambda - 1)^{n-1}$, we obtain the desired form. Also, $f(0) = (-1)^{n-1}$ and so the result for the sign holds as well.

Case 2. Suppose that X is not a tree. Since X is connected we conclude that X has at least n edges and thus contains at least one cycle C (See assignment 3, Question 6). Let e be any edge in the cycle C. Then, clearly, $X - e$ is still a connected graph. The graph X/e also is connected by the construction of the contraction. Finally, both of these graphs have strictly less than K edges (i.e. $K - 1$). Therefore, we can apply the induction hypothesis to those to find that

$$\begin{aligned} P_{X-e}(\lambda) &= \lambda f(\lambda) \\ P_{X/e}(\lambda) &= \lambda g(\lambda). \end{aligned}$$

with $f(0), g(0) \neq 0$. Then, by Theorem 5,

$$\begin{aligned} P_X(\lambda) &= P_{X-e}(\lambda) - P_{X/e}(\lambda) \\ &= \lambda(f(\lambda) - g(\lambda)). \end{aligned}$$

However, notice that $X - e$ has n vertices, whereas X/e has $n-1$ vertices. Therefore, $f(0)$ and $g(0)$ have opposite signs, and since neither of them are zero, we conclude that $f(0) - g(0) \neq 0$.

Finally, from Theorem 6, we know that

$$P_X(\lambda) = \lambda^n - |E(X)|\, \lambda^{n-1} + \cdots$$

with the coefficients alternating in sign. Notice then that $f(0) - g(0)$ is given by the coefficient of the term λ in the polynomial above. Since the terms alternate in sign, this term has sign $(-1)^{n-1}$. We then have all the intended results for X. □

We then have the following simple corollary.

Corollary 8. *Let X be a simple graph. Then, we can write its chromatic polynomial as $P_X(\lambda) = \lambda^c f(\lambda)$ where $f(0) \neq 0$ and c is the number of connected components of X.*

Proof. Let $X = \sqcup_{i=1}^{c} X_i$ where X_i are the connected components of X. Since all X_i's can be colored independently, it should be clear that

$$P_X(\lambda) = \prod_{i=1}^{c} P_{X_i}(\lambda).$$

Also, considering X_i as a subgraph, it is clearly a connected graph. Thus, by Theorem 7, we can write $P_{X_i}(\lambda) = \lambda f_i(\lambda)$ with $f_i(0) \neq 0$. We conclude that

$$P_X(\lambda) = \lambda^c \prod_{i=1}^{c} f_i(\lambda).$$

If we define $f(\lambda) = \prod_{i=1}^{c} f_i(\lambda)$, we get the desired result as $f(0)$ is a product of nonzero terms. □

Question 9

(a)

We begin by stating a few facts.

Proposition 9. *Let X be a simple graph and A its adjacency matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then,*

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{1}$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2\,|E(X)|\,. \tag{2}$$

Proof. Recall that $\mathrm{tr}(A^k) = \mathrm{tr}(U^{-1}A^kU) = \mathrm{tr}\left(\mathrm{Diag}(\lambda_1^k, \ldots, \lambda_n^k)\right)$ where U is the diagonalizing matrix. Also, we know that $\mathrm{tr}(A^k)$ corresponds to the number of closed paths of length k in X. Therefore, the sum in (1) above corresponds to the number of closed paths of length 1. Since X is loopless, this number is zero. The sum in (2) above corresponds to the number of closed paths of length 2. Since X is simple, these paths corresponds to going through a single edge twice, once in both direction. Since for every edge, we can start at either endpoint to do so, we conclude that there are $2\,|E(X)|$ of such paths. □

PROPOSITION 10. *If X is a simple graph with n vertices and e edges, its chromatic number $\chi(X)$ satisfies*

$$\chi(X) \leq 1 + \sqrt{\frac{2e(n-1)}{n}}.$$

Proof. First, recall Wilf's Theorem which states that

$$\chi(X) \leq 1 + \lambda_{\max}(X) \tag{3}$$

where $\lambda_{\max}(X)$ is the maximal eigenvalue of X. Without loss of generality, call the eigenvalues of X $\lambda_1, \ldots, \lambda_n$ (with multiplicity) and let $\lambda_1 = \lambda_{\max}$.

From equation (1) in Proposition 9, we know that

$$\lambda_{\max} = -\sum_{i=2}^{n} \lambda_i$$

and thus

$$\lambda_{\max}^2 = \left|\sum_{i=2}^{n} \lambda_i\right|^2.$$

By Cauchy-Schwarz inequality, we have

$$\lambda_{\max}^2 \leq \left(\sum_{i=2}^{n} |\lambda_i|^2\right)(n-1)$$

and by slightly rearranging equation (2) of Proposition 9, we replace the sum above to get

$$\begin{aligned}
&= (2e - \lambda_{\max}^2)(n-1) \\
&= 2e(n-1) - (n-1)\lambda_{\max}^2 \\
\Rightarrow n\lambda_{\max}^2 &\leq 2e(n-1) \\
\Rightarrow \lambda_{\max}^2 &\leq \frac{2e(n-1)}{n} \\
\Rightarrow \lambda_{\max} &\leq \sqrt{\frac{2e(n-1)}{n}}.
\end{aligned}$$

Replacing in the statement of Wilf's Theorem, we obtain

$$\chi(X) \leq 1 + \lambda_{\max}(X) \leq 1 + \sqrt{\frac{2e(n-1)}{n}}.$$

$\square$

(b)

PROPOSITION 11. *If X is a simple graph with e edges, its chromatic number $\chi(X)$ satisfies*

$$\chi(X) \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}} = \frac{1 + \sqrt{8e+1}}{2}.$$

Proof. Suppose that $\chi(X) = t$. Then, there is a way to color X using t colors. In particular, we can split the vertices of X into t color classes, i.e. $V(X) = \sqcup_{i=1}^{t} V_i$ where V_i are the vertices taking the color i. Given any two color classes, say V_i and V_j, notice that there needs to be an edge from V_i to V_j (more precisely from a vertex in V_i to a vertex in V_j). Therefore, grouping the vertices by their color, the graph *looks like* a complete graph on t vertices (if each color class plays the role of a vertex, there is an edge between any two of them). We conclude that there are at least $\frac{t(t-1)}{2}$ edges in the graph X, and therefore

$$\begin{aligned} \frac{t(t-1)}{2} &\leq e \\ t^2 - t &\leq 2e \\ t^2 - t + \frac{1}{4} &\leq 2e + \frac{1}{4} \\ \left(t - \frac{1}{2}\right)^2 &\leq 2e + \frac{1}{4} \\ t &\leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}. \end{aligned}$$

We conclude that $t = \chi(X) \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$. $\square$

Question 10

Let X_n be the graph with vertex set $\{1, 2, \ldots, 2n\}$ and where two vertices i and j are adjacent if and only if i and j share a prime factor.

PROPOSITION 12. *The chromatic number of X_n satisfies $\chi(X_n) \geq n$.*

Proof. Consider in X_n the vertices $\{2, 4, \dots, 2n\}$, i.e. the multiples of 2. There are n such vertices in X_n. Moreover, as 2 is a prime factor of all of them, we conclude that this set of vertices form a n-clique in X_n, that is the subgraph induced by those vertices is K_n. Any proper coloring of X_n needs to stay proper when restricted to this subgraph. Therefore, since we need at least n colors to color these vertices, we conclude that $\chi(X_n)$ is at least n. $\square$