Math 382: Solutions to Midterm test
Monday November 6, 2017; 11.35am to 12.15pm
No calculators or other electronic gadgets are allowed.

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1. Prove that $\sqrt{6}$ and $\sqrt{2}+\sqrt{3}$ are irrational numbers.
   Solution. If $\sqrt{6}$ were rational and equal to a/b (say), with a, b coprime, then we find upon squaring that $6b^2 = a^2$. As 2 divides the left hand side, it divides a^2 which means a is even. Thus, the right hand side is divisible by 4. But then, $4|6b^2$ means that $2|b^2$, contrary to the coprimality of a and b. Hence $\sqrt{6}$ is irrational. For the second part, if $\sqrt{2}+\sqrt{3}$ is rational, then so would its square. But then the square is $2+3+2\sqrt{6}$ which is irrational by the first part of the question.

2. (a) State Wilson's theorem and use it to show that if p is a prime ≥ 3, then $2(p-3)!+1$ is divisible by p.
   Solution. Wilson's theorem states that if p is a prime, then $(p-1)! \equiv -1 \pmod{p}$. Now $(p-1)! \equiv (p-3)!(p-2)(p-1) \pmod{p}$. But $(p-2)(p-1) \equiv 2 \pmod{p}$ so we deduce $2(p-3)! \equiv -1 \pmod{p}$, as required.

   (b) More generally, show that for any natural number k and a prime $p \geq k$,
   p divides
   $$(p-k)!(k-1)! + (-1)^{k-1}.$$
   Solution. By Wilson's theorem,
   $$-1 \equiv (p-1)! \equiv (p-k)!(p-k+1)(p-k+2)\cdots(p-1) \pmod{p}$$
   and noting that
   $$(p-k+1)(p-k+2)\cdots(p-1) \equiv (-1)^{k-1}(k-1)! \pmod{p}$$
   we deduce the desired result.