1. Prove that \(\sqrt{6}\) and \(\sqrt{2} + \sqrt{3}\) are irrational numbers.

**Solution.** If \(\sqrt{6}\) were rational and equal to \(a/b\) (say), with \(a, b\) coprime, then we find upon squaring that \(6b^2 = a^2\). As 2 divides the left hand side, it divides \(a^2\) which means \(a\) is even. Thus, the right hand side is divisible by 4. But then, \(4|b^2\) means that \(2|b^2\), contrary to the coprimality of \(a\) and \(b\). Hence \(\sqrt{6}\) is irrational. For the second part, if \(\sqrt{2} + \sqrt{3}\) is rational, then so would its square. But then the square is \(2 + 3 + 2\sqrt{6}\) which is irrational by the first part of the question.

2. (a) State Wilson’s theorem and use it to show that if \(p\) is a prime \(\geq 3\), then \(2(p - 3)! + 1\) is divisible by \(p\).

**Solution.** Wilson’s theorem states that if \(p\) is a prime, then \((p-1)! \equiv -1 \pmod{p}\). Now \((p-1)! \equiv (p-3)!(p-2)(p-1) \pmod{p}\). But \((p-2)(p-1) \equiv 2 \pmod{p}\) so we deduce \(2(p-3)! \equiv -1 \pmod{p}\), as required.

(b) More generally, show that for any natural number \(k\) and a prime \(p \geq k\),
\[
p \text{ divides } (p-k)!(k-1)! + (-1)^{k-1}.
\]

**Solution.** By Wilson’s theorem,
\[
-1 \equiv (p - 1)! \equiv (p - k)!(p - k + 1)(p - k + 2) \cdots (p - 1) \pmod{p}
\]
and noting that
\[
(p - k + 1)(p - k + 2) \cdots (p - 1) \equiv (-1)^{k-1}(k - 1)! \pmod{p}
\]
we deduce the desired result.