Math 402/802: Midterm Test

Students in Math 402: Do any two questions.

Students in Math 802: Do all questions.

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1. Let $a_0, a_1, a_2, ...$ be a sequence of numbers satisfying $a_0 = 1$, $a_1 = 3$ and

$$a_n = 2a_{n-1} + a_{n-2}, \qquad \text{for} \quad n \geq 2.$$

Set

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(a) Prove that

$$f(x) = \frac{1+x}{1-2x-x^2}.$$

(b) Show that

$$a_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right).$$

2. In a recent election, an opinion poll reported that the percentage of voters who would be satisfied with each of the three candidates T, H, M for prime minister was 65%, 57%, 58% respectively. Further, 28% would accept T or H, 30% would accept T or M, 27% would accept H or M and only 1% said they would accept any of the three. What percentage would accept none of them?

3. Let μ denote the classical Möbius function and F, G be real-valued functions defined on the positive reals.

(a) Prove that

$$G(x) = \sum_{n \leq x} F\left(\frac{x}{n}\right) \Leftrightarrow F(x) = \sum_{n \leq x} \mu(n) G\left(\frac{x}{n}\right),$$

where the summations are over positive integers $n \leq x$.

(b) Prove that

$$\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1.$$

Deduce that

$$\left|\sum_{n \leq x} \frac{\mu(n)}{n}\right| \leq 1.$$