VARIATIONAL STRUCTURE OF THE ZONES OF INSTABILITY

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Abstract. We give a variational characterization of the regions of stability and instability for planar Hamiltonian systems which are periodic in the time dependence, and convex in the momentum variables. In particular, we give necessary and sufficient conditions for stability, in terms of the Morse index of the variational problem with periodic or anti-periodic boundary conditions. We also show that this index coincides with the Maslov index of a curve of one-dimensional lines in the space $\mathbb{R}^2$. This result generalizes a result of Poincaré(1892) on unstable closed geodesics on orientable surfaces which minimize the arclength functional, and also gives a new test for stability based on the variational description of parametric resonance equivalent to that given by Liapunov(1892).

1. Introduction

In many physical systems described by a dynamical system of equations, loss of stability of an equilibrium can occur by varying the parameters in the system. For conservative systems this phenomenon is associated with parametric resonance. Instabilities are created when the frequency of an underlying stable periodic motion is in resonance with the frequency of a periodic fluctuation. A simple physical model of this is afforded by a oscillating pendulum whose length is changing periodically. A famous result due to Lord Rayleigh(1887) [15], states that the first resonance occurs when the frequency of the varying length is twice that of the underlying oscillation of the pendulum. The original impetus to study such systems came from the problem in celestial mechanics connected with determining the mean motion of the lunar perigee, Hill(1886) [8]. The important results in the subject were already found by Mathieu(1868), Floquet(1883), Hill(1886), Rayleigh(1887) and Liapunov(1892).

A convenient mathematical model for parametric resonance is afforded by the Mathieu equation, to which we shall refer, rather than stating here the most general case we consider.

The Mathieu equation is

$$\frac{d^2y}{dt^2} + (\lambda + 2\epsilon cos(2t))\, y = 0 \tag{1}$$

We are interested in the determination of the regions (of parameter space) which correspond to stable or unstable behavior of the solutions. Recall that this system is said to be stable if all solutions are bounded for all time. If some solutions are unbounded, the system (1) is said to be unstable. As is the case with the Mathieu equation (1), the system often depends on a parameter ϵ so that for convenience we associate the system (1) to a point in the (λ, ϵ) plane. We can then enquire about the regions of the parameter plane which correspond to stable or unstable behavior. These regions are separated by curves in the (λ, ϵ) plane which separate the two types of behavior [7]. Such curves are commonly called transition curves. The classical approach to studying these transitions, due originally to Liapunov [10], is to use the Floquet theory which yields a necessary and sufficient condition for strong stability, namely that the inequality hold

$$|\text{trace} R(\pi)| < 2 \tag{2}$$

where the fundamental matrix of normalized solutions is denoted by $R(t)$,

$$R(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix},$$

and the system has coefficients which are π periodic. On the boundary of the regions of strong stability, that is to say on the transition curves, we have the condition $|\text{trace} R(\pi)| = 2$. The transition curves have most frequently been studied using perturbation methods [6].

On the other hand, conservative systems can be associated with a variational principle. The geometry of the solutions can best be described using symplectic geometry. In fact, the matrix $R(t)$ represents a curve in the symplectic group $Sp(2, \mathbb{R})$ which starts at the identidy when $t = 0$. Our main result will give a condition, equivalent to Liapunov's, stated in terms of the Maslov index, which describes the rotation of Lagrange planes and their intersection with the vertical plane. The transition curves, in this formulation, will coincide with the moments of intersection of selected Lagrangian planes. The advantage to our approach, beyond the methodological one of using symplectic methods, is that our approach gives a global geometrical property of the system which can easily be transferred to the nonlinear setting. This method was used by Poincaré(1892) in his studies of closed geodesics on two dimensional surfaces. Indeed, our result is a direct generalization of a result of Poincaré which states that a nondegenerate closed geodesic on an orientable surface which minimizes arclength, must be hyperbolic. A minimizing geodesic has Morse index=0, and a corollary of our result states that any nondegenerate geodesic whose Morse index is even is hyperbolic. Such an approach has also been used, for example, to prove the elliptic stability of minimizers of the dual action for convex Hamiltonian systems of fixed energy, see Ekeland [5]. The existence of homoclinic orbits in the nonlinear forced pendulum equation [13] was proven with a version of Poincaré's theorem. This approach is also used in the study of elliptic and hyperbolic geodesics on two dimensional Riemannian manifolds [9]. However the results which are stated in [9] are incomplete and give a necessary conditon for ellipticity only. Our result gives necessary and sufficient conditions, and moreover is more general.

A second advantage, equally important, is the fact that the method has generalizations to higher dimensional settings, see [12, 14, 5]. The results in higher dimensions are not as complete as in the planar case, and indeed a complete description of the regions of stability would require careful consideration of the more challenging aspect of Krein indices, such as appears in [2, 5]. However a conjecture of G.D.Birkhoff [3] concerning the instability of minimizing closed curves in higher dimension has been answered [14]. Nevertheless, the planar problem offers a striking example of the method of variational analysis and symplectic geometry applied in the setting of classical results on stability.

The importance of periodic solutions to the determination of the regions of instability has long been known. This is evident for the Mathieu equation (1) for which the regions of instability are relatively well understood. These regions bifurcate from the transition values $\lambda = n^2$ along the $\epsilon = 0$ axis. The transition values correspond to eigenvalues of the periodic boundary value problem

$$\mathcal{L}y = \lambda y, \qquad y(0) = y(\pi), \qquad y'(0) = y'(\pi) \tag{3}$$

where we have used the differential operator $\mathcal{L}$,

$$\mathcal{L}y = -y'' - 2\epsilon cos(2t)y.$$

This operator is essentially self adjoint on the space of periodic functions $y(t)$ in $L^2[0,\pi]$. For the variational problem however, the natural space to consider are the periodic functions which are absoulutely continuous with derivative $\dot{y}(t)$ defined almost everywhere and which belong to $L^2[0,T]$. In the course of our discussion we shall denote this space by $H^1(S^1)$ to indicate that we are working with a standard Hilbert space of functions on the circle $S^1 = \mathbb{R}\mathrm{mod}2\pi$. The operator $\mathcal{L}$ with periodic boundary conditions has a spectrum of real eigenvalues

$$\mu_1 \leq \mu_2 \leq \mu_3 ... \to +\infty.$$

The Morse Index of the variational problem connected to the operator $\mathcal{L}$ with periodic boundary conditions , denoted $\mathrm{index}_{\Lambda^+}\mathcal{L}$, is equivalent to the number of such eigenvalues which are negative. As we shall see, the Morse index can best be understood and computed in terms of this variational problem for the function space $H^1(S^1)$.

Our first result gives a necessary condition for stability of the equation $\mathcal{L} = 0$ in terms of the $\mathrm{index}_{\Lambda^+}\mathcal{L}$. Similar results for geodesics on surfaces are reported in [9].

Theorem 1.1. *If the system (1) is stable (at $\lambda = 0$) then $index_{\Lambda^+}\mathcal{L}$ must be an odd positive integer.*

Next we consider an analogous boundary value problem but with anti-periodic boundary conditions. We obtain a corresponding necessary condition for stability.

Theorem 1.2. *If the system (1) is stable (at $\lambda = 0$) then $index_{\Lambda^-}\mathcal{L}$ must be an even positive integer.*

Our final result shows that these two theorems are in fact sufficient for stability as well.

For higher dimensional problems, that is when $n > 1$, there is much less that can be stated for linear systems with periodic coefficients in these simple terms. However we will also mention a result in this case which requires a certain symmetry condition on the coefficient matrix [12]. Here the corresponding system is

$$y'' + (\lambda I + Q(t))y = 0 \tag{4}$$

where λ is a spectral parameter, I denotes the $n \times n$ identidy matrix and $Q(t)$ is a symmetric $n \times n$ matrix with periodic coefficients.

Theorem 1.3. *If the matrix $Q(t)$ is even and periodic, and $index_{\Lambda^+}\mathcal{L} = 0$ then the system (4) at $\lambda = 0$ is unstable.*

2. Index Theorem

We begin by stating the range of problems to which our techniques will apply. This will be the class of linear Hamiltonian equations with periodic coefficients

$$\dot{x} = \frac{\partial H}{\partial y}, \quad -\dot{y} = \frac{\partial H}{\partial x}, \tag{5}$$

where $H = H(x, y, t) = H(x, y, t + T)$, and $(x, y) \in \mathbb{R}^2$. We assume throughout that H is quadratic in the phase variables (x, y) and satisfies the condition

$$\frac{\partial^2 H}{\partial y^2} > 0. \tag{6}$$

Under this assumption, the equations (5) are equivalent to Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \tag{7}$$

where the Lagrangian is defined classically via the Legendre transform

$$L(x, v, t) = \max_y [yv - H(x, y, t)] = Yv - H(x, Y, t),$$

$$\text{where} \quad v = \frac{\partial H(x, Y, t)}{\partial y}.$$

We also recall that the last equation is equivalent to

$$y(t) = \frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}}$$

for corresponding solutions of equations (5) and (7).

Due to the quadratic property of the Lagrangian, it is easy to see that

$$2L(x, \dot{x}, t) = \dot{x}\frac{\partial L}{\partial \dot{x}} + x\frac{\partial L}{\partial x}, \tag{8}$$

which will be used below to define the eigenvalues of the variational problem. These eigenvalues will coincide with the eigenvalues for the differential operator,

$$\mathcal{L} = [-D\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x}],$$

corresponding to Euler equations (7). Such eigenvalues will play a fundamental role.

We continue to use the notation $H^1(S^1)$ to denote the space of absolutely continuous functions on the circle $S^1 = \mathbb{R}(\mathrm{mod} T)$ which is endowed with the norm

$$\|x\|_1 = \int_0^T |x|^2 + |\dot{x}|^2 dt.$$

Functions belonging to $H^1(S^1)$ will satisfy the periodic boundary conditions $x(0) = x(T)$. We will also need to consider functions on the covering space of the circle but which satisfy anti-periodic boundary conditions, $x(0) = -x(T)$. For the sake of (abuse of) notation, we shall denote the space of such functions with the same norm, as $H^1(S^1_-)$.

As mentioned in the introduction, we shall introduce a variational problem on the space $H^1(S^1)$ which is connected with the differential operator $\mathcal{L}$. For this purpose we introduce the functional $A(x)$ on the space $H^1(S^1)$ defined by

$$A(x) = \int_0^T L(x, \dot{x}, t) dt,$$

where the Lagrangian L is calculated from the Legendre transform of the Hamiltonian H. Using the property (8) there is an associated bilinear form , which is symmetric on $H^1(S^1)$

$$A(\xi, \eta) = \frac{1}{2} \int_0^T \eta \frac{\partial L(\xi, \dot{\xi}, t)}{\partial \xi} + \dot{\eta} \frac{\partial L(\xi, \dot{\xi}, t)}{\partial \dot{\xi}} dt,$$

such that $A(x) = A(x, x)$. In addition we will use the symmetric bilinear functional on $H^1(S^1)$,

$$K(\xi, \eta) = \frac{1}{2} \int_0^T \eta \xi dt$$

A function $\xi \in H^1(S^1)$ and a real number μ are called an eigenpair for the symmetric bilinear form $A(\cdot, \cdot)$, relative to K, if for every $\eta \in H^1(S^1)$,

$$A(\xi, \eta) = \mu K(\xi, \eta).$$

A standard variational argument shows that the eigenvalues of the functional A on the space $H^1(S^1)$, coincide with the eigenvalues of the differential operator $\mathcal{L}$ on the space $L^2[0,T]$ with periodic boundary conditions (3).

In the following, we shall denote solutions of equation (5) by $(x(t), y(t))$ and for the corresponding space of initial conditions , we will write

$$(x(0), y(0)) \in \mathbb{R}^2.$$

Frequent use will be made of the isomorphism between the space of solutions to equation (5) and $\mathbb{R}^2$.

Definition 2.1. *The space of periodic solutions of equation (5) is denoted*

$$\Lambda^+ = \{(x(t), y(t)| \quad x(t) = x(t+T), \quad y(t) = y(t+T)\},$$

while the space of anti-periodic solutions will be denoted

$$\Lambda^- = \{(x(t), y(t))| \quad x(t) = -x(t+T), \quad y(t) = -y(t+T).\}$$

We define the nullity of the operator $\mathcal{L}$, with periodic boundary conditions, as the dimension of the eigenspace corresponding to the eigenvalue 0. We make a similar definition for anti-periodic boundary conditions. To describe the Index of the variational problem we are working with, in terms of periodic boundary conditions, we can use either the eigenspaces of the variational problem, or the corresponding eigenspaces of the differential operator $\mathcal{L}$. We define the Morse Index and the nullity of the operator $\mathcal{L}$ with periodic boundary conditions

$$\text{index}_{\Lambda^+}\mathcal{L} = dim\{ \text{ negative eigenspaces of } \mathcal{L}\}$$

$$\text{nullity}^+\mathcal{L} = dim\{ \text{ zero eigenspaces of } \mathcal{L}\}.$$

It is well known that the $\text{index}_{\Lambda^+}\mathcal{L}$ and the nullity of $\mathcal{L}$ are finite when condition (6) holds.

Theorem 2.1. *The null space of $\mathcal{L}$ with periodic boundary conditions is naturally isomorphic with the space of periodic solutions,*

$$nullity^{+}\mathcal{L} = dim\Lambda^{+}.$$

Moreover, the null space of $\mathcal{L}$ with anti-periodic boundary conditions is isomorphic with the space of anti-periodic solutions

$$nullity^{-}\mathcal{L} = dim\Lambda^{-}.$$

In order to describe the stability properties of $\mathcal{L}$, we introduce the following subspaces of solutions to equation (5).

$$W_{+} = \{(x,y)|\ \ x(0) = x(T)\}. \tag{9}$$

$$W_{-} = \{(x,y)|\ \ x(0) = -x(T)\}. \tag{10}$$

Functions $x(t)$ which belong to W_{+}, though not necessarily periodic, when restricted to the interval $[0,T]$ belong to the domain $H^1(S^1)$ of the functional A. Similar considerations apply for functions which belong to W_{-}. This restriction will always be inferred when we apply the functional A to an element of $W_{\pm}$.

We will need to use the Poincaré mapping G_T, which advances solutions of (5) by the period T. It is an easy matter to see that we have

$$(G_T - id)W_{+} = V, \quad \text{and} \quad (G_T + id)W_{-} = V,$$

where V denotes the vertical space

$$V = \{(0,y)|y \in \mathbb{R}\},$$

and id denotes the identidy map on $\mathbb{R}^2$.

Proposition 2.1. $\Lambda^{+} \subset W_{+}$, *and if* $\Lambda^{+} = 0$, *then*

$$W_{+} = (G_T - id)^{-1}V.$$

Moreover, $\Lambda^- \subset W_-$*, and if* $\Lambda^- = 0$*, then*

$$W_- = (G_T + id)^{-1} V.$$

Proof: It is clear that W_+ contains all periodic solutions of equation (5). If $\Lambda^+ = 0$, then G_T has no eigenvalue equal to one. Therefore $(G_T - id)$ is invertible and the statement of the Proposition follows from the alternate version of equation (9). Exactly the same argument works for the subspace W_-.

In order to state the classical Morse index theorem in our context, we introduce one more classical idea, that of the conjugate point.

Definition 2.2. *A point* $t_0 > 0$ *is conjugate to* 0 *if there exists a solution* $x(t)$ *of equation (7) with*

$$x(0) = x(t_0) = 0.$$

The conjugate index at t_0 *is the number of linearly independent solutions* $x(t)$ *which satisfy these boundary conditions.*

We state the classical Morse index theorem for periodic boundary conditions.

Theorem 2.2.

$$index_{\Lambda^+}\mathcal{L} = \sum_{0<t<T} conjugate\ \ index \quad + \quad concav_{\mathcal{L}}, \tag{11}$$

where the nonnegative integer $concav_{\mathcal{L}}$ *is defined by*

$$concav_{\mathcal{L}} \quad + \quad nullity^+ = dim\{(x,y) \in W_+ | A(x) \leq 0\}. \tag{12}$$

Now we can begin to formulate our first result as an application of the Morse index formula given above.

Before we do this however, it is useful to consider the computation of the term $\mathrm{concav}_{\mathcal{L}}$ in two situations which apply to the case when $\mathcal{L}$ is stable.

The first situation occurs when $\mathrm{nullity}^+\mathcal{L} = 2$. In this case $W_+ = \Lambda^+$ and it follows from equation (12) that $\mathrm{concav}_{\mathcal{L}} = 0$. If on the other hand, $\mathrm{nullity}^+\mathcal{L} = 0$, then $\mathrm{dim}W_+ = 1$, and

W_+ is spanned by (x, y) whose initial values are (u, v) where $G_T(u, v) = (u, v+1)$. Then according to equation (12) we have

$$\text{concav}_{\mathcal{L}} = \begin{cases} 0 & A(x) > 0 \\ 1 & A(x) \le 0 \end{cases} \tag{13}$$

Theorem 2.3. *If the system (5) is stable, then* $\text{index}_{\Lambda^+}\mathcal{L}$ *must be an odd positive number.*

Proof.

Let us assume that the system is strongly (elliptic) stable. This means the Liapunov multipliers of the monodromy operator G_T are complex conjugates which both belong to the unit circle, but not equal to $+1$. In particlar, this means that the nullity of $\mathcal{L}$ is zero. Later we shall consider the boundary case where the nullity is nonzero.

We will represent the Poincare mapping as the matrix

$$G_T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{where } \ ad - cb = 1, \quad \text{and} \quad a + d < 2. \tag{14}$$

We remark that the columns of G_T are formed by normalized solutions of equation (5) evaluated at the period T.

As we have mentioned earlier, W_+ is one-dimensional, and spanned by a solution of the matrix equation,

$$(G_T - I)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in W_+. \tag{15}$$

Using the matrix representative for G_T and simplifying, then applying the conditions on the coefficients stated in (14), we come to the equation

$$u(2 - a - d) = -b \tag{16}$$

We will also need to evaluate the term $\text{concav}_{\mathcal{L}}$. For this purpose, we will evaluate the functional $A(x)$, where (x, y) is a generator for W_+ whose intial conditons (u, v) satisfy the equation (15). Since $x(t)$ is a solution of the equation (7), then using the quadratic property

(8), after an integration by parts, we find

$$A(x) = (y(T) - y(0))x(0). \tag{17}$$

Upon using equation (15) to identify $x(0) = u = x(T)$, and $y(0) = v$, $y(T) = v + 1$, the exression (17) for $A(x)$ simplifies to

$$A(x) = u.$$

Now we let k denote the conjugate index, that is the number of conjugate points, $0 < t < T$, counted with their multiplicity. The second column of the matrix (14) is directly connected with k since $(b, d) = (x(T), y(T))$, where $x(0) = 0$.

In particular, if k is even, then upon a moments reflection we see that necessarily $b \geq 0$. This implies, by virtue of equation (16), that $u \leq 0$. Upon consulting equation (13), we infer that $\text{concav}_{\mathcal{L}} = 1$, and hence (equation (11)) $\text{index}_{\Lambda^+}\mathcal{L} = k + 1$ which is odd.

On the other hand, if k is odd, then necessarily $b \leq 0$. This implies from equation (16) that $u > 0$, and hence that $\text{concav}_{\mathcal{L}} = 0$. From this it follows that $\text{index}_{\Lambda^+}\mathcal{L} = k$ which is odd.

We remark that if $\text{nullity}^+\mathcal{L} = 1$, the system $\mathcal{L} = 0$ is unstable, since G_T is not semisimple, $\|G_T^n\|$ grows with the integer n. Consequently, there will be unbounded solutions of the equation $\mathcal{L} = 0$. The remaining case to consider therefore is the case when $\text{nullity}^+\mathcal{L} = 2$, which implies that every solution is periodic with period T. In this case, by virtue of equation (13), $\text{concav}_{\mathcal{L}} = 0$. However, if the conjugate index k is even, then $G_T(0,1) \neq (0,1)$ which contradicts the assumption that $\text{nullity}^+\mathcal{L} = 2$. Therefore, in this case, k must be odd and hence $\text{index}_{\Lambda^+}\mathcal{L} = k$ is odd as stated. This completes the proof.

3. Maslov Index

We now turn our attention to the case of anti-periodic boundary conditions and a further necessary condition for stability of the equation $\mathcal{L} = 0$. We will also describe the connection between the Morse index, discussed in the last section, and the Maslov index discussed below.

Lagrangian subspaces of a symplectic vector space are maximal subspaces on which the symplectic form vanishes. The Maslov index describes the intersection number of a curve of Lagrangian subspaces with a fixed Lagrangian subspace of a symplectic vector space [4]. Some care must be taken in computing this intersection number because, in general, intersections can occur with a positive or a negative sign. A curve which always has a positive intersection will be called a *plus curve.* In our setting, the symplectic space is $\mathbb{R}^2$, and the symplectic form is the standard signed area form in the plane, which for vectors $U, V \in \mathbb{R}^2$ can be calculated as the signed length of the cross product $U \times V$, positive for pairs of positively oriented vectors

$$\omega(U,V) = \pm\|U \times V\| = det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

Two elements of the space are orthogonal with respect to the symplectic form if $\omega(U,V) = 0$. Lagrangian subspaces of a symplectic vector space are maximal subspaces of self-orthogonal elements. Naturally, in our circumstance, this means that Lagrangian subspaces are just one dimensional subspaces in $\mathbb{R}^2$. The Maslov index of a closed curve (of lines) is just the rotation number of this curve, or equivalently the degree (rotation number) of a continuous function on S^1. We are interested in a special function on S^1 whose degree measures exactly the Morse index.

For this purpose, we consider the curve of lines through the origin in $\mathbb{R}^2$ described by the parameterization

$$t \longrightarrow W_+(t)$$

where the one dimensional subspace $W_+(t)$ is defined analogous to equation (9),

$$W_+(t) = \{(x,y)|x(0) = x(t)\}.$$

Let us note that if t is not conjugate to 0, then a standard argument shows that $W_+(t)$ is indeed one dimensional. On the other hand, if t is conjugate to 0, we can define $W_+(t) = \{(x,y)|x(0) = x(t) = 0\}$. We can investigate the intersections of the curve $W_+(t)$ with the vertical distribution $V_t = \{(0,y)|y \in \mathbb{R}\}$.

Our first observation is that nonzero intersections $W_+(t) \cap V_t$ occur only if t is conjugate to 0. Next we can show that the curve is a plus curve. For this purpose, we need a local parameterization of the space of lines in $\mathbb{R}^2$. The parameterization which is most natural from the point of symplectic geometry, is to represent each line L in an open set of such lines by a quadratic form on a given line in this space. This is equivalent to representing each line by a single real eigenvalue. As the parameter t varies over the period, we get a corresponding function on S^1. It is this function whose rotation number will be equivalent to the Morse index with periodic boundary conditions.

In a neighborhood of the line $W_+(t)$, the parameterization is accomplished by representing each nearby line L as the graph of a linear function on $W_+(t)$, $L = \text{graph } C_L$. Then the quadratic form on $W_+(t)$ which represents L, denoted $Q_+(L)$, can be evaluated using the symplectic form

$$Q_+(L)(U) = \omega(C_L(U), U), \quad U \in W_+(t)$$

This parameterization is described in detail in [4, 14]. Here, we merely notice that the quadratic form on $W_+(t)$, which describes the vertical space V can be described by

(18)

$$Q_+(V)\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \omega\begin{pmatrix} 0 & x(0) \\ y(0)-y(t) & y(0) \end{pmatrix} = x(0)(y(t)-y(0)) = x(t)(y(t)-y(0))$$

This quadratic form can be used to understand the intersections $W_+(t) \cap V_t$. At a conjugate point t_0, evidently, the eigenvalue is zero. The curve $W_+(t)$ is a plus curve if this eigenvalue crosses zero from negative to positive. Therefore, to show that $W_+(t)$ is a plus curve, we need to check the derivative, when $t = t_0$, of the expression (18)

$$\frac{d}{dt}Q_+(V)\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x'(t_0)y(t_0) = x'(t_0)\frac{\partial L}{\partial x'}(t_0).$$

We can now recognize that this term will always be positive, due to the convexity assumption on the Hamiltonian (6). This shows that indeed the curve $W_+(t)$ is a plus curve.

Theorem 3.1. *The Morse index described by equation (11) is equivalent to the Maslov index of the curve $W_+(t)$, $0 < t \leq T$.*

Proof Due to the explanation we have just given, concerning the fact that the curve $W_+(t)$ is a plus curve, counting intersections of $W_+(t) \cap V_t$ will involve counting the conjugate index. The Maslov index will be increased by one each time t passes through a conjugate point. However the curve $W_+(t)$ is not closed in the space of one-dimensional Lagrangian lines. This problem is rectified by first joining the endpoints of the curve $W_+(t)$, and computing the Maslov index of the resulting closed curve. It is always possible to join these endpoints with a segment whose intersection number with the vertical line is equal to the index of $Q_+(V)$ evaluated at the endpoint [4]. Therefore there needs to be added a boundary term which accounts for the fact that the curve is not closed. This correction term must account for whether the quadratic form Q_+ is postive or nonpositive when evaluated at the endpoint of the curve which of course is just $W_+(T)$. To compute the index, we notice that the quadratic form Q in expression (18) may be evaluated using the variational integral, namely

$$A(x) = Q_+(V)\begin{pmatrix} x(T) \\ y(T) \end{pmatrix}$$

which follows upon an integration by parts, since $x(t)$ is a solution of the equation (7). Therefore, it follows that the index and nullity of the quadratic form Q_+, evaluated at the endpoint of the curve $W_+(T)$, will be equal to the quantity which appears in the determination of the concavity (12):

$$\text{index} + \text{nullity } Q_(V) + \begin{pmatrix} x(T) \\ y(T) \end{pmatrix} = \{(x, y) \in W_+ | A(x) \leq 0\}.$$

This explains the correction term which appears in the Morse index formula, as an special case of the computation of the Maslov index, and completes the proof of the equivalence of this index with the Morse index for periodic boundary conditions.

We are now ready to consider the differential operator $\mathcal{L}$ with anti-periodic boundary conditions. Our goal is to represent the number of negative eigenvalues as the Maslov index of a curve of lines, and to show how this Maslov index can be computed. This computation will then be used to give a second necessary condition for stability. The eigenvalue problem we consider for the operator $\mathcal{L}$ is

$$\mathcal{L}x = \lambda y, \qquad x(0) = -x(T), \qquad x'(0) = -x'(T). \tag{19}$$

This boundary value problem is associated with a variational principle on the space $H^1(S^1_-)$, as the periodic boundary value problem (3) was associated with a variational problem on the space $H^1(S^1)$. This variational problem will consider critical values of the functional A,

$$A(x) = \int_0^T L(x, x', t)dt, \qquad x \in H^1(S^1_-)$$

To describe the nullity of the variational problem, we consider the case of zero eigenvalues, for the boundary value problem $\mathcal{L}y = 0$, with anti-periodic boundary conditions (19). The dimension of the eigenspace for eigenvalue 0 will be equal to the nullity of the variational problem on the space $H^1(S^1_-)$.

Proposition 3.1. *The solutions of the boundary value problem (19) with $\lambda = 0$, correspond one to one with elements of the subspace of anti-periodic solutions Λ^- of the equations (5). Thus the number of zero eigenvalues is equal to the dimension of Λ_-*

$$nullity^- \mathcal{L} = dim \Lambda^-.$$

We now want to describe the index of the operator $\mathcal{L}$ with anti-periodic boundary conditions. This will be equal to the number of negative eigenvalues λ for the boundary value problem (19). This is also equal to the maximal dimension of the subspace of $H^1(S^1_-)$ on which the functional A is negative definite. As we did with the periodic problem, we can associate this Morse index with a Maslov index for a curve of one dimensional subspaces of $\mathbb{R}^2$. The one dimensional subspace of solutions of the equation (5) will be given by the subspace

$$W_-(t) = \{(x, y) | x(0) = -x(t)\}.$$

This subspace will change as the parameter t goes from 0 to T. At the endpoint of this interval, the subspace coincides with the space W_- see equation (10). The Maslov index of this curve can be computed as it was in the periodic case. We first need to check that this curve is a plus curve in the sense described above. For this purpose we introduce the quadratic form on the subspace $W_-(t)$, analogous to the case of (18),

(20)
$$Q_-(V)\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \omega\begin{pmatrix} 0 & x(0) \\ y(0)+y(t) & y(0) \end{pmatrix} = -x(0)(y(t)+y(0)) = x(t)(y(t)+y(0))$$

To check that the curve $W_-(t)$ is a plus curve, we need to differentiate this equation with respect to t, when $t = t_0$ is conjugate to 0. A calculation similar to the one for the quadratic form Q_+ shows that, at a point conjugate to 0,

$$\frac{d}{dt}Q_-(V)\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x'(t_0)y(t_0) = x'(t_0)\frac{\partial L}{\partial x'}(t_0) > 0.$$

Thus we have proven,

Lemma 3.1. *The curves $W_+(t)$, and $W_-(t)$, $0 < t \leq T$, are plus curves.*

The determination of the index of the functional A, on the subspace $H^1(S^1_-)$ can now be related to the Maslov index of the curve $W_-(t)$ over the interval $0 < t \leq T$, as we did in the case with periodic boundary conditions. In the following we have taken the liberty of defining a new term, called the convexity of $\mathcal{L}$, which is completely analogous to the term concavity which was coined by Morse.

Theorem 3.2.

$$index_{\Lambda^-}\mathcal{L} = \sum_{0<t<T} conjugate \;\; index \;\; + \;\; convex_{\mathcal{L}}, \tag{21}$$

where the nonnegative integer $convex_{\mathcal{L}}$ is defined by

$$convex_{\mathcal{L}} \;\; + \;\; nullity_- = dim\{(x,y) \in W_- | A(x) \leq 0\}. \tag{22}$$

This theorem will now be used to prove the next necessary condition for the stability of the system (5). As we did for the periodic case, we first consider the computation of the term $\text{convex}_{\mathcal{L}}$ in two situations which apply to the case when $\mathcal{L}$ is stable.

The first situation occurs when $\text{nullity}\_\mathcal{L} = 2$. In this case $W_- = \Lambda^-$ and it follows from equation (22) that $\text{convex}_{\mathcal{L}} = 0$. If on the other hand, $\text{nullity}\_\mathcal{L} = 0$, then $\dim W_- = 1$, and W_- is spanned by (x, y) whose initial values are (u, v) where $G_T(u, v) = (-u, -v - 1)$. Then according to equation (22) we have

$$\text{convex}_{\mathcal{L}} = \begin{cases} 0 & A(x) > 0 \\ 1 & A(x) \le 0 \end{cases} \tag{23}$$

Theorem 3.3. *If the system (5) is stable, then* $\text{index}_{\Lambda^-}\mathcal{L}$ *must be an even number.*

Proof.

As before, let us assume first that the system is stongly or elliptic stable. Then we have $\text{nullity}\_\mathcal{L} = 0$ so that W_- is one dimensional and generated by (x, y), where $(x(0), y(0)) = (u, v)$, and

$$(G_T + I)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in W_-. \tag{24}$$

Using the representation (14) for G_T and writing the components of equation (24) we get

$$(a+1)u + bv = 0,$$

$$\text{and} \qquad cu + (d+1)v = -1,$$

which can be simplified by using the fact that $(ad - cb) = 1$, to arrive at

$$u(2 + a + d) = b. \tag{25}$$

We will use the fact that $(2 + a + d) > 0$, since G_T is elliptic stable. Now, we can calculate $A(x)$, for $(x, y) \in W_-$ with an integration by parts,

$$A(x) = x(T)(y(T) + y(0)) = Q_-\begin{pmatrix} x(T) \\ y(T) \end{pmatrix}.$$

Since we have chosen a special generator (x, y) for W_-, we may substitute,

$$x(T) = -u, \text{ and } \quad y(T) = -1 - v$$

so that $A(x) = u$.

Now we let k denote the conjugate index on the interval $0 < t < T$. If k is odd and positive, then $b = u(2 + d + a) < 0$ which implies that $u < 0$ and $\text{convex}_{\mathcal{L}} = 1$ by virtue of equation (23). Thus in this case, equation (21) implies that $\text{index}_{\Lambda^-}\mathcal{L} = k + 1$ which is even.

If, on the other hand, k is even and non-negative, then $b = u(2+a+d) > 0$ which implies that $u > 0$ which in turn implies that $\text{convex}_{\mathcal{L}} = 0$. Therefore, appealing again to equation (21), we find that $\text{index}_{\Lambda^-}\mathcal{L} = k$ which is even.

The other case which must be considered is when nullity_$\mathcal{L} = 2$, and k is the conjugate index $0 < t < T$. However, in this case, every vector (u, v) is anti-periodic under the mapping G_T and in particular

$$G_T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

which is impossible if k is odd. Therefore k is even in this case as well. This completes the proof that stability implies that $\text{index}_{\Lambda^-}\mathcal{L}$ must be an even integer.

4. Test for Stability

We have shown in Theorem (2.3) and Theorem(3.3) that a necessary condition for stability is that

$$\text{index}_{\Lambda^-}\mathcal{L} \text{ is even, and } \text{index}_{\Lambda^+}\mathcal{L} \text{ is odd.}$$

We will show that these conditions are sufficient for stability as well. First however, let us state a corollary of the previous results, which gives a sufficient condition for instability, and which generalizes the result of Poincaré(1892) on hyperbolic minimizing closed geodesics. This is just the restatement of the necessary condition for stability.

Corollary 4.1. *If either* $\text{index}_{\Lambda^-}\mathcal{L}$ *is odd, or* $\text{index}_{\Lambda^+}\mathcal{L}$ *is even, then the system (5) is unstable.*

To prove the sufficiency test for stability, we will prove the logical negation of the sufficient condition for instability

Theorem 4.1. *If the system (5) is unstable, then either* $index_{\Lambda^-}\mathcal{L}$ *is odd, or* $index_{\Lambda^+}\mathcal{L}$ *is even.*

Proof

If the system (5) is unstable, then we have in case G_T is semisimple,

$$(26) \qquad G_T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{where } ad - cb = 1, \quad \text{and either } a + d < -2, \text{ or } a + d > 2.$$

The remaining case, will be when G_T has a nilpotent part, and in this case either

$$\text{nullity}^+\mathcal{L} = dim\Lambda^+ = 1, \text{ or}$$

$$\text{nullity}^-\mathcal{L} = dim\Lambda^- = 1.$$

We will work with the semisimple case first. The proof will be broken down into two parts, corresponding to the cases where either $a + d < -2$, or $a + d > 2$.

In the first case, by choosing a generator (x, y) of the subspace W_- whose intial values are (u, v) with $G_T(u, v) = (-u, -v - 1)$, we arrive as before at the equation (25). Let k denote the conjugate index, $0 < t < T$. If k is odd, then $u(2 + d + a) = b < 0$ which implies in the first case, that $u > 0 = A(x)$. By virtue of equation (22), this means that $\text{convex}_{\mathcal{L}} = 0$ and $\text{index}_{\Lambda^-}\mathcal{L} = k$ which is odd.

If k is even, then $u(2+d+a) = b > 0$ which implies that $A(x) = u < 0$, and $\text{convex}_{\mathcal{L}} = 1$. Therefore, $\text{index}_{\Lambda^-}\mathcal{L} = k + 1$ which is odd.

Now we assume on the contrary that the second case holds, $a + d > 2$. In this case, we choose a generator (x, y) of W_+ with intial values (u, v) which satisfy $G_T(u, v) = (u, v + 1)$. As in equation (15) this leads to equation (16), $u(2 - a - d) = -b$. In case k is even, $u(2 - a - d) = -b \leq 0$, which means that $A(x) = u > 0$ and $\text{concav}_{\mathcal{L}} = 0$. Therefore, $\text{index}_{\Lambda^+}\mathcal{L} = k$ is even. If the index k is odd, then $u(2 - a - d) = -b > 0$ which means that

$A(x) = u < 0$. Therefore $\text{concav}_{\mathcal{L}} = 1$ which implies from equation (21) that $\text{index}_{\Lambda^+}\mathcal{L} = k + 1$ which is even.

Thus, in case G_T is semisimple and unstable, then either $\text{index}_{\Lambda^+}\mathcal{L}$ is even, or $\text{index}_{\Lambda^-}\mathcal{L}$ is odd. The remaining case to consider is when G_T is not diagonalizeable, and either dim $\Lambda^+ = 1$, or dim $\Lambda^- = 1$.

We will analyze the case when $\Lambda^- = 1$, the other case being similar. We assume that

$$G_T = \begin{bmatrix} -1 & 0 \\ c & -1 \end{bmatrix}$$

which implies that W_- is two dimensional. By choosing a generator (x, y) of the subspace W_- whose intial values are (u, v) with $G_T(u, v) = (-u, -v - 1)$, we calculate, as before that $A(x) = u$. Now we consider the conjugate index k. If W_- is two dimensional, then a moments reflection will convince us that k must be even. Now we can calculate the term $\text{convex}_{\mathcal{L}}$. This will be $+1$ since $A(x) = u < 0$. Thus the index in this case will be $k + 1$ which is odd as stated. This completes the proof of the theorem.

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