Mathematics 227 - Vector Analysis. Introduction
Instructor: Daniel Offin
Jeffrey Hall 408
Webpage://WWW. mast.queensu.ca /˜offind/ math227

Examples of flow rates and flux

- Calgary, June 2013, Bow and Elbow rivers catastrophic flooding in and around Calgary Alberta. On CBC it is announced that flow rate is extremely high 19.5 $\frac{\text{meters}}{\text{second}}$. What does this flow rate measure?

- ¡ Fukushima nuclear reactors in Japan. Levels of nuclear radiation from contaminated groundwater 18 times higher than previously measured. Radiation flux measured in *grays* or *millisieverts*. 1 gray $= \frac{\text{joule}}{\text{second}}$. Radioactive and charged particles are influenced by magnetic fields, so since magnetic flux is higher near the poles, effective dose increases in higher lattitudes. How is this flow rate measured?

- Brightness of stars measured by light flux density. The inverse square law states that this is proportional to the inverse square of the distance from the star. How is this measured?

- Flux of magnetic field lines near the magnetic poles of the earth higher than at equator. Measurement?

- heat flux density (rate of heat flow per unit time per unit area)

- current density (charge per unit time per unit area)

Meaurement of flow of physical quantity in space

- light flux density is light energy per unit time, per unit area (electromagnetic energy measured in watts per unit area).

$$W = \frac{\text{joule}}{\text{second}} = \frac{N \cdot m}{s} = \frac{\text{kg} \cdot m^2}{s^3}$$

  Luminosity (measured in watts) is electromagnetic energy radiated per unit time. To measure the luminosity of a light bulb, put a hollow sphere around the light bulb, and measure the energy absorbed on the surface of the sphere. The light flux (brightness) and the luminosity L of the light bulb are related by

$$\begin{aligned} F &= \frac{L}{\text{Area}}, \quad \text{brightness } \frac{\text{watts}}{m^2} \\ &= \frac{L}{4\pi R^2}, \quad \frac{W}{m^2}. \end{aligned}$$

  Surface area of a hollow sphere is $A = 4\pi R^2$. This describes the brightness of stars, as the inverse square law of distance, once you determine luminosity L.

- The key ideas contained in all these examples, is **flux density** of some physical quantity which is being transported in space. This quantity could be **water, light energy, heat, radioactivity, electrical charge, chemical concentration**. To measure this density, we record how the quantity Δq passes through some surface $\mathbf{S}$ in space, in time Δt. This surface has small elements of area A, and we define density by letting this area element tend to zero.

$$J = \lim_{A \to 0} \frac{I}{A} = \frac{dI}{dA}$$

  where J is the flux density, and I is the **rate** at which the **transport of quantitiy** q occurs across a surface $\mathbf{S}$ with area element $\mathbf{A}$

$$I = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t}$$

- To determine **total flux**, we need to add all the components of the flux density, across the various area elements A_i of an entire surface $\mathbf{S}$ in space (composed of many small sections or elements).

$$J = \sum_i J_i \Delta A_i, \quad \text{as } \Delta A_i \to 0.$$

  This is a familiar operation in calculus, that of taking a **Riemann sum** , which we learn about in single or multivariable calculus. But in this case, now we need to understand how to make such a computation over an arbitrary surface $\mathbf{S}$.

- The total flux as we have defined it now reprsents the **flow rate per unit time**, of the physical quantity q, across a surface $\mathbf{S}$ in space. To repeat what we said earlier, this flow rate could be water or chemical in solution, being transported. It could also be energy transported per unit time (light, or radioactive energy), or heat energy transported across a surface, per unit time. How would we present this surface and how can we learn how to make such computations?

- In the case of the flooding in Calgary, we can look again at what the CBC announced as flow rate, and try to better understand what was being described by the rate 19.5 $\frac{m}{s}$. In this case, the physical quantity $\mathbf{q}$ represents the quantity of water transported across a boundary surface in the river. This quantity $\mathbf{q}$ is a volume, and so is measured in cubic meters m^3. The flow rate density, measures the amount of water which crosses this surface per unit time, per unit area! This is the **Flux density** of which we spoke earlier, measured now in cubic meters per second per square meter or $\frac{m^3}{s \cdot m^2} = \frac{m}{s}$.

- This simple dimensional analysis tells us that what the CBC announcer was trying to convey was a **flux density** of water crossing a boundary surface $\mathbf{S}$ in the Bow river. This boundary surface is of course nothing more than a crossection of the river. So to calculate total volume of water crossing per second, we need to integrate the density over the cross section $\mathbf{S}$ of the river. In this example, we suppose that the flux density really is an average density, over the surface $\mathbf{S}$.
  How would we define an average density over a surface? How would we compute it?

- Amoung many questions which are raised by such considerations, we mention one question which arises: does it matter which crossection $\mathbf{S}$ we choose to make this computation? Apparently not, if between the two crossections $\mathbf{S}$ and $\mathbf{S}'$ there are no sources (inlets of additional water) then the flux across each surface $\mathbf{S}$ and $\mathbf{S}'$ would be the same.

- The **physical velocity** of a particle of water in the river is given by a **vector field** which is a vector quantity $\tilde{\mathbf{V}}$ having magnitude (speed) and direction at each point. This mathematical entity is one of the basic objects we will study in this course. We assume that the student has familiarity with the basic notions of vector algebra (addition, scalar multiplication) and the related ideas of geometry in Euclidean space.

- For example we will be able to test the vector field $\tilde{\mathbf{V}}$ itself, to determine whether there are any sources (of water, light, radioactivity etc.) between two crossections $\mathbf{S}, \mathbf{S}'$. We will prove a result called the divergence theorem, for general vector fields $\tilde{\mathbf{V}}$ and sources between general complete surfaces $\mathbf{S}, \mathbf{S}'$. This theorem has fundamental applications in diverse areas of scientific investigation from electromagnetic disturbance to transport of volume through space.

- We will begin our course with discussions concerning integrals over geometric regions, and vector operations.

Multiple integrals

Before we begin our investigation of vector fields, we review the basic definition, that of the concept of definite integral over a rectangle in the plane $\mathbb{R}^2$.

The rectangle $\mathbf{R} \subset \mathbb{R}^2$ is denoted

$$\mathbf{R} = \{(x, y) | a \leq x \leq b, \quad c \leq y \leq d\}$$

If we subdivide this rectangle into subrectangles $A_{i,y}$ which are determined by subdividing the intervals defining the rectangle $\mathbf{R}$

$$a = x_0 < x_1 < \ldots < x_n = b, \quad c = y_0 < y_1 < \ldots < y_m = d$$

The double integral of the function $f(x,y)$ over the rectangle $\mathbf{R}$ is

$$\int\int_{\mathbf{R}} f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{i=n}\sum_{j=1}^{j=m} f(x^*_{i,j}, y^*_{i,j})\Delta A$$

whenever this limit exists. The function f is called **integrable** when this limit exists. We can choose the points $(x^*_{i,j}, y^*_{i,j})$ arbitrarily within the $i-j^{th}$ rectangle $A_{i,j}$.
For example we could choose the points within $A_{i,j}$ to coincide with the upper right hand point so that in this case $(x^*_{i,j}, y^*_{i,j}) = (x_i, y_j)$.

Applications

- Volume beneath the graph of f, and lying over the region $\mathbf{R}$ equals the double integral $\int\int_{\mathbf{R}} f(x,y)dA$, whenever $f \geq 0$, througout $\mathbf{R}$.

- when f changes sign over the region $\mathbf{R}$, we can speak of the signed volume
$$\int\int_{\mathbf{R}} f(x,y)dA = \int\int_{\mathbf{R_1}} f(x,y)dA - \int\int_{\mathbf{R_2}} f(x,y)dA = V_1 - V_2,$$
where V_1 is the positive volume over the region $\mathbf{R_1} = \{(x,y)|f(x,y) \geq 0\}$, and V_2 is the negative volume over the region $\mathbf{R_2} = \{(x,y)|f(x,y) \leq 0\}$.

- **Average Value**. When f is integrable over the region $\mathbf{R}$, we have
$$\int\int_{\mathbf{R}} f(x,y)dA = f_{\text{avg}} \times \text{area } \mathbf{R}.$$

Properties of the definite integral

- $\int\int_{\mathbf{R}}(f(x,y) + g(x,y))dA = \int\int_{\mathbf{R}} f(x,y)dA + \int\int_{\mathbf{R}} g(x,y)dA$

- $\int\int_{\mathbf{R}} cf(x,y)dA = c\int\int_{\mathbf{R}} f(x,y)dA$

- $\int\int_{\mathbf{R}} f(x,y)dA \geq \int\int_{\mathbf{R}} g(x,y)dA$ whenever $f(x,y) \geq g(x,y)$ throughout the region $\mathbf{R}$.