SOLUTIONS

Name and Student Number

Math/MTHE 280, Advanced Calculus, Fall 2016
Queen's University, Department of Mathematics
Please write your student number and your name clearly at the top of this page.

Additional space for calculations can be arranged on the back of each page or on additional blank pages at the end of the exam.
Do all five questions, marks are indicated. Total marks are 54.

1a) . **[5 marks]** Is the function f(x,y) continuous for all values of $(x, y) \in \mathbb{R}^2$? Explain your answer.

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{\sqrt{x^2+2y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Using polar coordinates

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta - r^3\sin^3\theta}{\sqrt{r^2\cos^2\theta + 2r^2\sin^2\theta}} = r^2\frac{[\cos^3\theta - \sin^3\theta]}{\sqrt{\cos^2\theta + 2\sin^2\theta}}$$

denominator $= \sqrt{1+\sin^2\theta} \geq 1$.

$$-r^2 \cdot 2 \leq f(r,\theta) \leq r^2(2) \quad \text{therefore}$$

$$\lim_{r\to 0} f(x,y) = 0 = f(0,0)$$

1b) . **[5 marks]** Is the function $f(x, y)$ from part a) differentiable at (0,0)?. Explain your answer.

We need to consider the linear approximation at $(0,0)$

$$H(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{h^3}{h|h|} = 0 \qquad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{-h^3}{h|h|\sqrt{2}} = 0$$

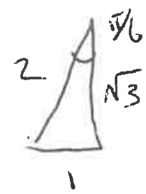
Thus, $H(x,y) = 0$. We consider whether this is a good approximation

$$\lim_{(x,y)\to(0,0)} \left|\frac{f(x,y) - H(x,y)}{\sqrt{x^2+y^2}}\right| = \left|\frac{x^3-y^3}{\sqrt{x^2+y^2}\sqrt{x^2+2y^2}}\right| \leq \frac{x^3+y^3}{x^2+y^2} \to 0$$

Differentiable at $(0,0)$

2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the change of coordinates

$$F(r,\theta) = (r\cos(\theta), r\sin(\theta)), \qquad F\left(2, \frac{\pi}{6}\right) = (\sqrt{3}, 1)$$



Suppose that $G : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, and denote $H = G \circ F$,

a)[4 marks] If $G_x(\sqrt{3}, 1) = -1$, $G_y(\sqrt{3}, 1) = +3$, calculate $DH\left(2, \frac{\pi}{6}\right)$.

$$DG(\sqrt{3},1) = (-1, 3), \quad DF = \begin{pmatrix} \cos\pi/6 & -2\sin\pi/6 \\ \sin\pi/6 & 2\cos\pi/6 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1 \\ 1/2 & \sqrt{3} \end{pmatrix}$$

By chain rule,

$$DH(2, \pi/6) = DG(\sqrt{3},1)\, DF(2,\pi/6)$$

$$= (-1, 3)\begin{bmatrix} \sqrt{3}/2 & -1 \\ 1/2 & \sqrt{3} \end{bmatrix} = \left(\frac{3}{2} - \frac{\sqrt{3}}{2},\ 1 + 3\sqrt{3}\right)$$

b) [6marks] If $H_r\left(2, \frac{\pi}{6}\right) = 0$ and $H_\theta\left(2, \frac{\pi}{6}\right) = 2$ calculate $DG(\sqrt{3}, 1)$.

$$[DF]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/4 & \sqrt{3}/4 \end{bmatrix}$$

$$DH(2,\pi/6) = DG(\sqrt{3},1) \cdot DF(2,\pi/6)$$

$$DH(2,\pi/6)\,(DF)^{-1} = DG(\sqrt{3},1)$$

$$[0, 2]\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/4 & \sqrt{3}/4 \end{bmatrix} = DG(\sqrt{3},1) = \left[-\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

3a).[4 marks] Consider the three dimensional surface $\mathbf{S} = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 8\}$. Find the equation of the tangent plane to the surface **S** at the point $(-1, \sqrt{3}, -\sqrt{3}, +1)$

If $F(x,y,z,w) = x^2 + y^2 + z^2 + w^2$, then surface S is the level set $\{F = 8\} = S$.

$\nabla F = (2x, 2y, 2z, 2w) = (-2, 2\sqrt{3}, -2\sqrt{3}, 2)$

is perpendicular to the tangent plane at $(-1, \sqrt{3}, -\sqrt{3}, 1)$

Equation of tangent plane is $0 = \nabla F \cdot \vec{V}$, or

$-2(x+1) + 2\sqrt{3}(y-\sqrt{3}) - 2\sqrt{3}(z+\sqrt{3}) + 2(w-1) = 0$.

b)[3 marks] Show that the parameterized circle $\vec{\mathbf{r}}(t) = (-1, \sqrt{6}\sin(t), -\sqrt{6}\cos(t), +1)$ is tangent to the surface **S** at the parameter value $t = \frac{\pi}{4}$.

At parameter value $t = \pi/4$, $\frac{d\vec{r}}{dt} = \left(0, \frac{\sqrt{6}}{\sqrt{2}}, \frac{\sqrt{6}}{\sqrt{2}}, 0\right)$

and $\frac{d\vec{r}}{dt} \cdot \nabla F(-1, \sqrt{3}, -\sqrt{3}, 1)$

$= \frac{d\vec{r}}{dt} \cdot (-2, 2\sqrt{3}, -2\sqrt{3}, 2) = (2)(3) - (2)(3) = 0$

thus $\left.\frac{d\vec{r}}{dt}\right|_{t=\pi/4}$ is a tangent vector to S.

c)[3 marks] Using the parameterized curve $\vec{\mathbf{r}}(t)$ from **b)**, calculate the derivative of the composite function $f(\vec{\mathbf{r}}(t))$ at $t = \frac{\pi}{4}$ where $f(x, y, z, w) = \ln\left(\frac{w}{y}\right) + \ln\left(\frac{z}{x}\right)$.

By chain rule using $Df(x,y,z,w) = \left[-\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}, \frac{1}{w}\right]$

$\frac{d}{dt} f(\vec{r}(t)) = Df(\vec{r}(\pi/4)) \cdot \frac{d\vec{r}}{dt}(\pi/4)$

$= \left[-\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}, \frac{1}{w}\right] \cdot \left[0, \sqrt{3}, \sqrt{3}, 0\right]$

$= \cancel{\sqrt{3}\sqrt{6}} - \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{-\sqrt{3}} = -2$.

4. Let $\vec{F} = zy\vec{\mathbf{i}} + xz\vec{\mathbf{j}} + xy\vec{k}$

(a). [4 marks] Show that divergence and curl of $\vec{F}$ are both zero.

$$\operatorname{div} F = \frac{\partial}{\partial x}(zy) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0.$$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zy & xz & xy \end{vmatrix} = (x-x)i - (y-y)j + (z-z)k.$$

(b). [6 marks] Show that $\vec{F}$ is a gradient field, and compute the potential function $f(x, y, z)$ for which $\vec{F} = \nabla f$.

Since $\operatorname{curl} \vec{F} = 0$, we look for function $f(x,y,z)$ so that 1) $\frac{\partial f}{\partial x} = zy$, 2) $\frac{\partial f}{\partial y} = xz$, 3) $\frac{\partial f}{\partial z} = xy$.

Each of these can be integrated to give

1) $f(x,y,z) = zyx + h_1(z,y)$.
2) $f(x,y,z) = xzy + h_2(x,z)$
3) $f(x,y,z) = xyz + h_3(x,y)$.

where h_1, h_2, h_3 are ~~constants~~ functions of partial integration.

We see that choosing $h_1 = h_2 = h_3 = 0$ gives

$f(x,y,z) = xyz$. and $\boxed{\nabla f = F}$

(c) [2 marks] Which surface **S** in $\mathbf{R}^3$ is $\vec{F}$ orthogonal to at the point (1,1,1)?

The level set of the function f constructed in part b), which includes the point $(1,1,1)$

$$S = \{(x,y,z) \mid xyz = 1\}.$$

Then $\vec{F} = \nabla f$ is orthogonal at $(1,1,1)$ to S.

5 Consider the vector field $\vec{\mathbf{F}}(x,y,z) = 4y\vec{i} - x\vec{j}$.
(a)[4 marks]Show that $\vec{\mathbf{F}} = (x\vec{i} + 4y\vec{j}) \times \vec{k}$ and $\nabla \times \vec{\mathbf{F}} = -5\vec{k}$

$$\boxed{\begin{aligned} i \times k &= -j \\ j \times k &= i \end{aligned}}$$

$$(xi + 4yj) \times k = -xj + 4yi = \vec{F}(x,y).$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y & -x & 0 \end{vmatrix} = 0i - 0j - (1+4)$$

b) [4 marks] Show that $\vec{r}(t) = \left(a\sin(2t), \frac{a}{2}\cos(2t), 0\right)$ is a flow line for the vector field $\vec{\mathbf{F}}$ for every value of the parameter a. What geometric curves are described by these parameterized paths. Sketch a few of them.

$$\frac{d\vec{r}}{dt} = (2a\cos 2t, -a\sin 2t, 0), \text{ and}$$

$$\vec{F}(\vec{r}(t)) = \left[4\left(\tfrac{a}{2}\right)\cos 2t, -a\sin 2t\right]$$

We see immediately that

$\frac{d\vec{r}}{dt} = \vec{F}(r(t))$ for each parameter t, and every value a. (sketch next page)

c) [4 marks] Prove that $\vec{\mathbf{F}}$ is not a gradient field, that is no such function $f(x,y,z)$ exists so that $\vec{\mathbf{F}} = \nabla f$ **You may use the following blank page for additional writing space if necessary.**

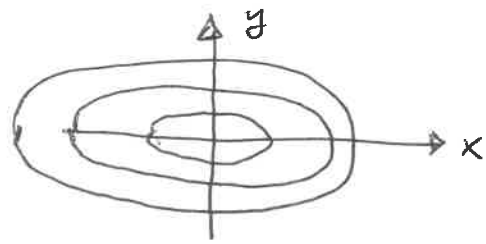
Since curl $\vec{F} = -5k \neq 0$, $\vec{F}$ cannot be a gradient field.

Extra page for recording work and answers. Please indicate carefully which questions you are adding material here for.

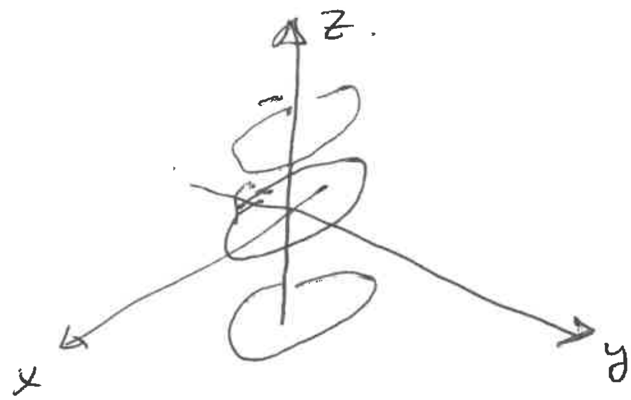
sketch of some flow lines for $\vec{F} = 4y\hat{i} - x\hat{j}$

The curves are all ellipses (geometric description)

since $x^2 + 2y^2 = a^2$.



planar



spatial

etc.