The following exercises are from *Vector calculus* (3rd ed.) by Colley:

Exercise 1 (1.1.26 pg. 8). *A 50 kg sandbag is suspended by two ropes. Suppose that a three-dimensional coordinate system is introduced so that the sandbag is at the origin and the ropes are anchored at the points* $(0, 2, 1)$ *and* $(0, -2, 1)$.

a). Assuming that the force due to gravity points parallel to the vector $(0, 0, -1)$, *give a vector* $\mathbf{F}$ *which describes the gravitational force.*

b). Now, use vectors to describe the forces along each of the two ropes.

Solution. a). By assumption, the force $\mathbf{F}$ due to gravity points parallel to the vector $\mathbf{v} := (0, 0, -1)$, hence is proportional to $\mathbf{v}$, i.e.

$$\mathbf{F} = \lambda \mathbf{v},$$

where λ is some real number. Let's also assume that $\mathbf{F}$ points along the negative z-axis, so that $\lambda > 0$ (this doesn't follow from the statement of the problem, but it was probably intended). Since $\mathbf{v}$ has length one, λ is simply equal to the magnitude $\|\mathbf{F}\|$ of the force $\mathbf{F}$. By general physics, we have[1] $\|\mathbf{F}\| = mg$, where m is the mass of the sandbag and $g \sim 9.81 \frac{m}{s^2}$ is a constant of proportionality. Therefore,

$$\mathbf{F} = (0, 0, -mg) = (0, 0, -490.5N).$$

b). Let $\mathbf{T_1}$ and $\mathbf{T_2}$ be the tension forces in the ropes anchored at $(0, 2, 1)$ and $(0, -2, 1)$, respectively. Since the bag is stationary, we have

$$\mathbf{T_1} + \mathbf{T_2} + \mathbf{F} = 0.$$

By symmetry, the tension forces have no x-component. We can decompose the force vectors into components parallel to the y- and z-axis, writing

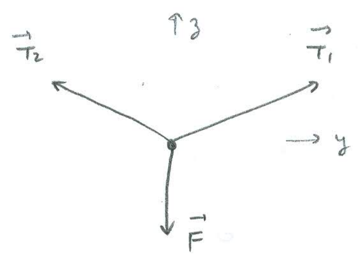
$$\mathbf{T_1} = (0, T_{1,y}, T_{1,z}), \qquad \mathbf{T_2} = (0, T_{2,y}, T_{2,z}), \qquad \mathbf{F} = (0, 0, -mg).$$

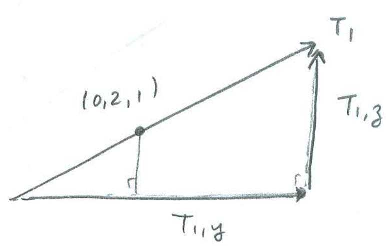
And, invoking symmetry again, we have $T_{1,y} = -T_{2,y}$ and $T_{1,z} = T_{2,z}$. Since also $T_{1,z} + T_{2,z} = mg$, it follows that $T_{1,z} = T_{2,z} = \frac{1}{2}mg$. We know that $\mathbf{T_1}$ points along $(0, 2, 1)$, so by similarity we have $T_{1,y} = 2T_{1,z}$. Alternatively, to see this we can do some quick trig: calling the angle that T_1 makes with the positive y-axis θ, we have

$$\frac{1}{2} = \tan(\theta) = \frac{T_{1,z}}{T_{1,y}}.$$

Therefore, $T_{1,y} = 2T_{1,z} = mg$.

Putting the above together, $\mathbf{T_1} = (0, mg, \frac{mg}{2})$ and, by symmetry, $\mathbf{T_2} = (0, -mg, \frac{mg}{2})$.





[1] Well, assuming that we're on Earth and sufficiently close to the surface.

Exercise 2 (1.3.24, pg. 26). *Let A, B, C and D be four points of $\mathbb{R}^3$ such that no three of them lie on a line. Then $ABCD$ is a quadrilateral (not necessarily one that lies in a plane). Let the midpoints of the four sides of $ABCD$ be denoted by M_1, M_2, M_3 and M_4. Use vectors to show that $M_1M_2M_3M_4$ is always a parallelogram.*

Solution. The statement that $M_1M_2M_3M_4$ is a parallelogram (i.e. a simple quadrilateral whose opposite sides are parallel) is equivalent to the statement that we have the vector equalities

$$\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3} \quad \text{and} \quad \overrightarrow{M_4M_1} = \overrightarrow{M_3M_2}.$$

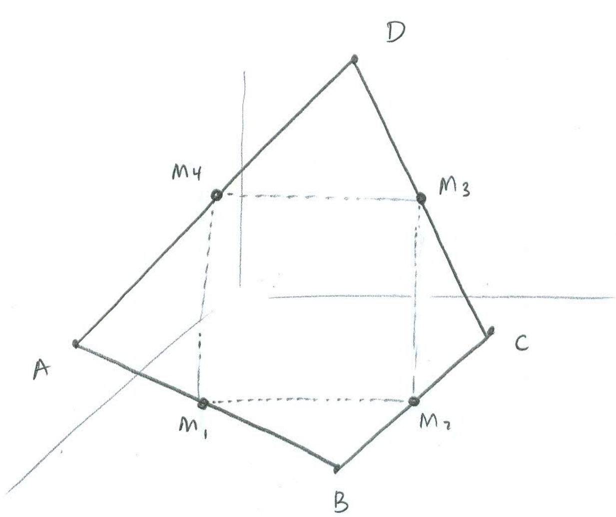
Now, looking at the figure below, we see that

$$\begin{aligned}
\overrightarrow{M_1M_2} &= \overrightarrow{M_1B} + \overrightarrow{BM_2} = \frac{\overrightarrow{AB} + \overrightarrow{BC}}{2}, \\
\overrightarrow{M_2M_3} &= \overrightarrow{M_2C} + \overrightarrow{CM_3} = \frac{\overrightarrow{BC} + \overrightarrow{CD}}{2}, \\
\overrightarrow{M_3M_4} &= \overrightarrow{M_3D} + \overrightarrow{DM_4} = \frac{\overrightarrow{CD} + \overrightarrow{DA}}{2}, \\
\overrightarrow{M_4M_1} &= \overrightarrow{M_4A} + \overrightarrow{AM_1} = \frac{\overrightarrow{DA} + \overrightarrow{AB}}{2}.
\end{aligned}$$

Then

$$\begin{aligned}
\overrightarrow{M_1M_2} - \overrightarrow{M_4M_3} &= \overrightarrow{M_1M_2} + \overrightarrow{M_3M_4} = \frac{\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}}{2} \quad \text{and} \\
\overrightarrow{M_4M_1} - \overrightarrow{M_3M_2} &= \overrightarrow{M_4M_1} + \overrightarrow{M_2M_3} = \frac{\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}}{2},
\end{aligned}$$

But $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$ is just the vector formed by going 'once around' $ABCD$, so it is equal to the 0 vector, as desired!



Exercise 3 (1.5.33). *Let Π be the plane in $\mathbb{R}^3$ with normal vector* $\mathbf{n}$ *that passes through the point* A *with position vector* $\mathbf{a}$. *If* $\mathbf{b}$ *is the position vector of a point* B *in* $\mathbb{R}^3$, *show that the distance between* B *and* Π *is given by*

$$D = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|}.$$

Solution. First, $(\mathbf{b} - \mathbf{a})$ is simply the vector $\overrightarrow{AB}$. Looking at the figure below and using some geometric intuition, it is clear that the distance from B to Π is given by the length of the projection of $\overrightarrow{AB}$ to the line spanned by $\mathbf{n}$. But

$$\text{proj}_{\mathbf{n}}(\mathbf{b} - \mathbf{a}) = \frac{\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})}{\|\mathbf{n}\|^2}\mathbf{n}.$$

by formula (5) on page 22, so that

$$D = \left\| \frac{\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})}{\|\mathbf{n}\|^2}\mathbf{n} \right\| = \left| \frac{\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})}{\|\mathbf{n}\|^2} \right| \|\mathbf{n}\| = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|}.$$