5.3 The soulberry patch

You have signed on to work 10 hours this weekend for Farmer Brown. Your plan is to dig turnips for which he has offered you a wage of $12 per hour.

But upon your Friday night arrival, he announces that the early spring has rendered his soulberry patch ripe for the picking, and he offers you a chance at that. “Awesome,” you reply. “How much?”

Now Farmer Brown is wise in the ways of the young and knows better than to offer you a flat hourly wage for picking anything as tempting as his prize soulberries—rather, he will pay per basket picked. “Same as last year,” he says. “$2 per basket.”

Well, you have two possibilities here—digging turnips and picking soulberries—and you have 10 hours available. Digging turnips (at $12/hour) will give you $120. That’s simple enough. But what will you get from picking soulberries? The difficulty here is that the payoff is given, not per hour but per basket. So what you need to know is how many baskets you will be able to pick per hour.

Well the situation here is just a little bit complicated. It turns out that your picking rate changes as you pick. At the beginning, when there are lots of berries hanging high on the vine, you can pick them at a high rate, but as the patch becomes picked over, the berries that remain are increasingly likely to be hiding their succulent charms under leaves and behind (gasp!) thorns, and your picking rate goes down. Fortunately, your early soulberry years (you grew up next door to this very farm) have given you an exceedingly precise measure of this picking rate, and you are now in possession of the remarkable graph reproduced at the right which displays the total number \( S \) of baskets you can harvest in the first \( x \) hours of picking.

Given this, how should you spend your 10 hours?—digging turnips or picking soulberries, or some of each?

(a) Using rate of change ideas, find a graphical argument for the optimal allocation. Use a construction on the graph to find your maximum total revenue.

(b) Remarkably, the \( S-x \) graph turns out to be a piece of a parabola with equation:

\[
S = \frac{x(24-x)}{2} \quad (0 \leq x \leq 10).
\]

Use this formula to check your graphical estimates in (a).
(a) Graphical analysis

This is something like the alligator egg problem in that we have two activities and we can compare our rate of return in each at any time. To start with we ought to draw the soulberry graph and the turnip graph on the same set of axes. There’s a problem here in comparing them—the soulberry return is measured in baskets and the turnip return in dollars. The way to handle this is to convert baskets to dollars (multiply by 2) and use dollars as the common vertical axis. The two graphs are plotted at the right. The turnip graph is a straight line of slope 12.

We can see that if you decided to spend your whole time in only one patch, the soulberry patch would be the place to be (getting $140 as opposed to $120). But what about splitting the time between the two patches?

The argument is similar to the alligator egg argument. At the beginning, the soulberry graph has the higher slope which means that the soulberry patch has the higher rate of return. So that’s where you should start. As time goes on, the slope and therefore the rate of return decreases. At some point, the graph becomes parallel to the turnip graph and the rate of return has fallen to $12/h. If you stay any longer the rate of return will fall below $12/h and you would have been better off in the turnip patch. So you ought to switch at the moment the rate falls to $12/h. From the graph, this is seen to occur at $x = 6$. You should spend the first 6 hours in the soulberry patch and then switch to turnips and stay there for the remaining 4 hours.

What is your total revenue from this allocation? Your 6 soulberry hours gives you almost $110 (read from the graph), say 108. The remaining 4 hours at $12/h gives you $48. Your total revenue is: $108 + 48 = 156$.

Again we can read this from the graph by taking off from the soulberry graph at $x = 6$ on a line of slope 12. The total revenues will be the height of this line after another 4 hours, that is, at $x = 10$, the right-hand edge of the graph. The agreement of this with our estimate above of $156 is extremely good.
(b) *Algebraic analysis.*  
Now we are given a formula for the soulberry graph and we are to use it to verify our estimate of the optimal switch point \( x \) and the total revenue \( R \).

If you pick soulberries for \( x \) hours and dig turnips for the remaining \( 10-x \) hours, your total revenue will be

\[
R(x) = \text{soulberry revenue} + \text{turnip revenue} \\
= 2S(x) + 12(10-x) \\
= 2 \frac{x(24-x)}{2} + 12(10-x) \\
= 24x - x^2 + 120 - 12x \\
= -x^2 + 12x + 120.
\]

There's our expression for \( R \). We want to choose the value of \( x \) which makes this a maximum.

This type of question can be hard to answer without the power of the calculus, but in this case \( R \) is a parabola and we know how to find maxima and minima for such a function. Since the coefficient of \( x^2 \) is negative, the parabola opens down and the graph will have a maximum at its vertex. The simplest way to find that is to strip off the constant term. We are left with

\[-x^2 + 12x\]

and this will have the same vertex. It can be factored as

\[-x^2 + 12x = -(x-12)\]

which displays the zeros at \( x=0 \) and \( x=12 \). For a parabola, the vertex is half-way between the zeros at \( x=6 \). Hence \( R \) takes its maximum at \( x=6 \) and that's the optimal point to switch from soulberries to turnips. Your maximum revenue will be

\[
R(6) = [-x^2 + 12x + 120]_{x=6} \\
= (-6)^2 + 12(6) + 120 \\
= 36 + 72 + 120 \\
= 228.
\]

just as we found in (a).
Problems

1. Here we look at the effect if we change the amount that Farmer Brown is prepared to pay per basket of soulberries. We assume the same relationship as before between the time $x$ you spend in the patch and the total number $S$ of baskets picked. We also assume the same turnip digging rate of $12$/h.

(a) Suppose Farmer Brown offers you only $1.60 per basket. This is a lower soulberry rate than the original $2.00 per basket, and our intuition tells us that you should spend less time in the soulberry patch than before. Using rate of change ideas, present a graphical argument for the optimal allocation. Use a construction on the graph to find your maximum total revenue.

For the remainder of this problem, use the soulberry basket formula

$$S(x) = \frac{x(24-x)}{2} \quad (0 \leq x \leq 10).$$

(b) Make an algebraic check of your graphical estimates in (a).

(c) Now let the soulberry rate be any value whatsoever, so we give it a variable name—$k$ dollars per basket. Find an expression for your optimal switch point $x$ in terms of $k$.

[The variable $k$ is called a parameter. In any particular situation it will have a fixed value (like 2 or 1.5) but by carrying it through the calculation as a symbol, we can solve a whole family of optimization problems in one go.]

(d) What soulberry rate would lead you to split your time equally between berry-picking and turnip digging?

(e) Our intuition tells us that the higher is the value $k$ of a basket of soulberries, the more time you should spend in the soulberry patch. Check this out.

(f) How high should $k$ be for you to want to spend your entire 10 hours picking soulberries?

2. Here we examine the effect of variations in the turnip digging rate. Assume the soulberry rate of the previous section: $2$ per basket.

(a) Assume a turnip digging wage of $15$/h. Using rate of change ideas, present a graphical argument for the optimal allocation. Use a construction on the graph to find your maximum total revenue.

For the remainder of this problem, use the soulberry basket formula

$$S(x) = \frac{x(24-x)}{2} \quad (0 \leq x \leq 10).$$

(b) Make an algebraic check of your graphical estimates in (a).

(c) Find a general formula for your optimal switch point $x$ in terms of the hourly turnip digging wage $d$.

(d) Our intuition tells us that the higher is $d$, the more time you should spend in the turnip patch. Check this out.

(e) How large would the turnip rate have to be for you to decide to spend your entire 10 hours digging turnips?

(f) How small would the turnip rate have to be for you to spend your entire 10 hours picking berries?
3. In each case you are given a formula for a variable $x$ in terms of a variable $k$ valid for $k > 0$. Now as we increase $k$ above 0, $x$ will change. In each case determine whether $x$ increases or decreases. Of course it’s possible that it does both, first one and then the other. There are a number of different possible strategies you might use here. It may be enough simply to examine the formula as given. You might want to rewrite it first, as we did above. Or you might want to think about what the graph of $x$ against $k$ looks like.

(a) $x = 10 - k$
(b) $x = k^2 - 1$
(c) $x = k^2 + 4k$
(d) $x = k(k-2)$
(e) $x = \frac{1}{k-1}$
(f) $x = \frac{1}{1 - \frac{1}{k}}$
(g) $x = \frac{(k+1)}{k}$
(h) $x = \frac{(k+6)}{k+1}$ [Hint. Change variable. Let $h = k+1$.]

4. In each of the eight parts of #3, above, find the value of $k$ which makes $x = 3$.

5. Wildberries! All set for another action packed weekend at Farmer Brown’s. The turnips are finished but the yams are ready to be dug and the wage for that is again $12 per hour. But this time it’s the wildberry patch that’s ready to be picked. Now wildberries are small and rare (and wild!) with a flavour to die for, and Farmer Brown will pay you $4 per basket. So you have the same old problem—how to allocate your 10 hours between the two activities to maximize your total revenue?

As before, to answer this question, you need to know the rate at which you can pick wildberries. Well that’s easy enough—you pick at a constant rate of 4 baskets per hour—the patch is large and you just roam. But there is a complication. The patch is off in the wild (surprise surprise) an hour’s drive away and if you decide to do any picking at all, that means a total of two hours’ driving time which has to count as part of your 10 hours. You can use Farmer Brown’s truck, so there’s no gas cost.

(a) Is it worth driving out to the wildberry patch and if so, how long should you spend picking? [Answer: yes. spend all 10 hours—2 driving and 8 picking.]

(b) Let Farmer Brown’s wildberry rate be $k$/basket. Find your optimal allocation in terms of $k$. [Answer: For $k < 3.75$, 10 hours digging. For $k > 3.75$, 2 driving and 8 picking.]
6. Your father is trying to persuade you to take a correspondence course in Latin this summer. Now you already have a cool job lined up constructing web pages at a wage of $10/h, and you can spend as much time as you want at that over the summer. So when your father invokes your allegiance to an ancient and noble language and reads you passages from Virgil, you simply shrug. “Very well,” he replies, fighting fire with fire, “I’ll offer you $10 for every mark that you obtain on the final exam.” Now there’s a proposition.

Of course, what you need to know is how time spent studying Latin can be expected to translate into marks. Fortunately, the situation with regard to summer courses in Latin has been extensively analyzed. For students of your ability, experience and dedication, a precise graphical relationship is known to hold between your expected course mark \( M \) and your investment \( x \) in hours. And by good luck, you just happen to have a copy of that graph stashed away in your drawer of “possibly useful things.” You note with some satisfaction that even if you spend no time on the course \( (x=0) \) you can expect to get 10 marks based on your intuition, your Roman heritage, and your childhood study of Winne-Ille-Pu. There’s 100 bucks right there. Hmm.

(a) What does the shape of the curve tell you about the way in which your study time will translate into marks. Is this reasonable?

(b) What you’d like is to get an equation for the curve. It turns out to be a piece of a parabola with vertex just off the graph at \( x=110 \). Another useful point is the mark of 90 that you could get with 100 hours of work. [No thanks.] And then there’s the gift of 10 marks with no work at all. Use these three pieces of information to find an equation for \( M \) in terms of \( x \).

[Answer: \( M = \frac{-x^2 + 220x + 1500}{150} \).]

(c) How many hours should you allocate to the Latin course in order to maximize your total revenue?

(d) Will the allocation of (c) be enough to give you a pass (50%)?

(e) Now we let your father’s payoff be \( $k \)/mark. Find a formula for the optimal study time \( x \) in terms of \( k \).

(f) What value of \( k \) in (e) would prescribe an investment of 50 hours of study?

(g) Our intuition tells us that the higher is \( k \) in (e), the more time you should devote to the Latin course. Check this out.

(h) Your father decides that he really wants you to get a “first” (80%). How much per mark will he have to offer you to accomplish that?